General Considerations

- Consider all components
  - Projectile motion (can simplify to 2D)
    + Ignoring air resistance $\ddot{\vec{x}} = -\vec{g} \Rightarrow \ddot{x} = 0,\ \ddot{z} = -g$

      so $x = x_0 + v_{x,0}t$, $z = z_0 + v_{z,0}t - \frac{1}{2}gt^2$.

      You can solve for $z(x)$. This is <u>parabolic motion</u>
    + With linear air resistance $\ddot{x} + \gamma\dot{x} = 0$, $\ddot{z} + \gamma\dot{z} = -g$ $(\gamma = \frac{\lambda}{m})$

      so $x = \frac{v_{x,0}}{\gamma}\left(1 - e^{-\gamma t}\right)$, $z = \left(\frac{v_{z,0}}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}$

      Then $$z = \frac{\gamma v_{z,0} + g}{\gamma v_{x,0}}\, x + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{v_{x,0}}\right)$$

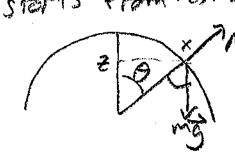
      The <u>range</u>, or distance in x before $z \to 0$, is given by a transcendental equation.
    + Quadratic air resistance has

      $$m\ddot{x} + \lambda\sqrt{\dot{x}^2 + \dot{z}^2}\,\dot{x} = 0, \quad m\ddot{z} + \lambda\sqrt{\dot{x}^2 + \dot{z}^2}\,\dot{z} = -g$$

      Requires numerical solution.
  - Circular motion: motion with constant $\vec{r}^2$.
    + First, $\frac{d}{dt}(\vec{r}^2) = 0 \Rightarrow \vec{r}\cdot\vec{v} = 0$, so $\vec{v} \perp \vec{r}$

      Next $\frac{d}{dt}(\vec{r}\cdot\vec{v}) = 0 \Rightarrow \vec{a}\cdot\vec{r} + v^2 = 0 \Rightarrow a_r = -\frac{v^2}{r}$ (<u>centripetal acceleration</u>)
    + <u>Example</u>: Suppose an object slides down a frictionless sphere of radius r. When does it fall off, if it starts from rest at the top?

      Newton's law in the radial direction is

      $$ma_r = -\frac{mv^2}{r} = -mg\cos\theta + N$$

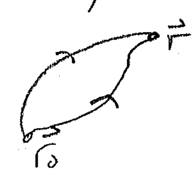
      By energy conservation, $\frac{1}{2}mv^2 + mgz = mgr$, so

      $$N = mg\cos\theta - \frac{1}{r}(2mgr - 2mgr\cos\theta) = mg(3\cos\theta - 2)$$

      The object leaves the sphere when $N = 0$, or when $\cos\theta = 2/3$.

– Energy + Conservation

- Consider as before kinetic energy $T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\dot{\vec{r}}^2$
  - \+ This implies
    $$\frac{d}{dt}(T) = m\vec{v}\cdot\dot{\vec{v}} = \vec{F}\cdot\vec{v} \text{ by } 2^{nd} \text{ Law}$$
  - \+ We see that a force $\perp\ \vec{v}$ (always) does no work
    This can include normal force while object moves along a surface
    Also includes the Lorentz force of magnetic field on charge $\vec{F} = q\vec{v}\times\vec{B}$.
  - \+ The work done moving from $\vec{r}_1$ to $\vec{r}_2$ is
    $$\Delta T = W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} \overset{?}{=} -V(\vec{r}_2) + V(\vec{r}_1) \text{ if we want a conserved total energy } E = T + V$$
  - \+ A <u>conservative force</u> can be defined as $\vec{F} = -\vec{\nabla}V$ for potential energy V. Fundamental theorem of calculus gives the above
- When is that possible?
  - \+ If $\vec{F} = -\vec{\nabla}V$, then we know $\vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\vec{\nabla}V = 0$
    That is, conservative forces have zero curl
  - \+ How do we define V? If we choose $V(\vec{r}_0) = 0$, then
    $$V(\vec{r}) \equiv -\int_{\vec{r}_0}^{\vec{r}} d\vec{r}\,'\cdot\vec{F} \text{ along some path}$$



  - \+ For this to be independent of path,
    $$\oint d\vec{r}\cdot\vec{F} = 0 \Rightarrow \int d\vec{S}\cdot(\vec{\nabla}\times\vec{F}) = 0 \quad \text{Check!}$$

– Angular Momentum

- We define the <u>angular momentum</u> of a particle as
  $$\vec{J} = \vec{r}\times\vec{p} \text{ with a given origin}$$
  - \+ In Cartesian components, $\vec{J} = m[(y\dot{z} - z\dot{y})\hat{\imath} + (z\dot{x} - x\dot{z})\hat{\jmath} + (x\dot{y} - y\dot{x})\hat{k}$

+ But in spherical coordinates, note $\hat{r}\times\hat{\theta}=\hat{\phi}$, $\hat{r}\times\hat{\phi}=-\hat{\theta}$,

so $$\vec{J} = mr^2\dot{\theta}\,\hat{\phi} - mr^2\sin\theta\,\dot{\phi}\,\hat{\theta}$$

+ That's why this is <u>angular</u> momentum

- Torque + Central forces
  + The time derivative is $\frac{d}{dt}(\vec{J}) = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}} = 0 + \vec{r}\times\vec{F}$
  + We define <u>torque</u> $\vec{\tau} = \vec{r}\times\vec{F}$, so $\dot{\vec{J}} = \vec{\tau}$

    Note: KBc calls torque $\Rightarrow$ "moment of force" and $\vec{\tau}\to\vec{G}$.
    Others may use $\vec{\tau}\to\vec{N}$ or $\vec{\Gamma}$.
  + A <u>central force</u> between two objects acts along $\vec{r}_1-\vec{r}_2$.
  + In many cases, the origin is on the line between $\vec{r}_1$ & $\vec{r}_2$,
    so a central force is along $\hat{r}$. We will typically consider
  + one object moving around a central force from the origin.

    In these cases, $\vec{\tau}=0 \Rightarrow$ <u>angular momentum</u> is <u>conserved</u>

- Torque on a composite object
  + For an object made of many particles, $\vec{J} = \sum_i \vec{r}_i\times\vec{p}_i$

    and $\dot{\vec{J}} = \sum_{i,j} \vec{r}_i\times(\vec{F}_{ij} + \vec{F}_{i,ext})$ where $\vec{F}_{ij}$ are internal forces
  + Let's look at the internal forces
    $$\sum_{i,j}\vec{r}_i\times\vec{F}_{ij} = \frac{1}{2}\sum_{i,j}(\vec{r}_i\times\vec{F}_{ij} + \vec{r}_j\times\vec{F}_{ji}) = \frac{1}{2}\sum_{i,j}(\vec{r}_i-\vec{r}_j)\times\vec{F}_{ij}$$
    For central forces, $\vec{F}_{ij}\parallel(\vec{r}_i-\vec{r}_j)$, so $=0$. Actually more general.
  + Therefore (very generally) $\dot{\vec{J}} = \sum_i \vec{r}_i\times\vec{F}_{i,ext} = \vec{\tau}_{ext}$.

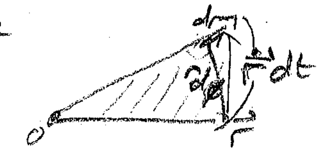
- Consequences of Angular Momentum Conservation (single object)
  + When $\vec{J}$ is conserved, motion is in a plane $\perp$ $\vec{J}$

+ The total energy is $E = T + V = \frac{1}{2}m\vec{v}^2 + V(\vec{r}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + V(r)$ (if we choose axes with $\vec{J}$ along $\hat{k}$, so $\theta = \pi/2$. This is

$$E = \frac{1}{2}m\dot{r}^2 + \left(V(r) + \frac{J^2}{2mr^2}\right),$$

so we have an <u>effective potential</u> $U(r) = V(r) + \frac{J^2}{2mr^2}$ for radial motion, 2nd term is "centrifugal"

+ Consider the area swept out by the vector from the origin to the object. For small enough dt, the area is that of the triangle with sides $\vec{r}$, $\dot{\vec{r}}dt$, $\vec{r}+\dot{\vec{r}}dt$,



or $dA = \frac{1}{2}(r+dr)(rd\phi) \approx \frac{1}{2}r^2 d\phi$. In other words, the area swept out per time $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{J}{2m}$ is constant.

This is <u>Kepler's 2nd law of planetary motion</u> but is generally true.

— Isotropic Harmonic Oscillator (An Example)

- A general harmonic oscillator in 3D has $V(\vec{r}) = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2 + \frac{1}{2}k_3z^2$
  + Let's consider the case with $k_1 = k_2 = k_3$, or $V = \frac{1}{2}kr^2$, $\vec{F} = -k\vec{r}$ is central.
  + This is <u>isotropic</u> — the same in every direction.
- Solution in Cartesian Coordinates
  + This is just harmonic oscillation in each dimension with the same frequency $\omega = \sqrt{k/m}$
  + Therefore, we have $\vec{r}(t) = \vec{r}_0\cos(\omega t) + (\vec{v}_0/\omega)\sin(\omega t)$
  + We know the motion must lie in a plane, and we see directly that it is in the plane spanned by $\vec{r}_0$ and $\vec{v}_0$. Let's choose this to be the x-y plane
  + Now note

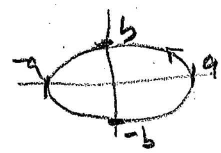
$$\vec{r}(t)\cdot\vec{v}(t) = \frac{1}{2}\left(\frac{v_0^2}{\omega} - \omega r_0^2\right)\sin(2\omega t) + (\vec{r}_0\cdot\vec{v}_0)\cos(2\omega t)$$

vanishes for $\tan(2\omega t) = \frac{2\omega\,\vec{r}_0\cdot\vec{v}_0}{\omega^2 r_0^2 - v_0^2}$, ie 4 times in 1 period.

We can shift t, to make $t=0$ one of these times $\Rightarrow \vec{r}_0 \cdot \vec{v}_0 = 0$.

+ By rotating axes, we can make $x(t=0) = r_0$ (the max value), $y(0)=0$ while $\dot{x}(0) = 0$, $\dot{y}(0) = v_0$. Then

$$x = r_0 \cos(\omega t) \qquad y = (v_0/\omega)\sin(\omega t)$$



+ This is motion in an ellipse: note that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a = r_0,\ b = v_0/\omega \text{ are major + minor semi-axes.}$$

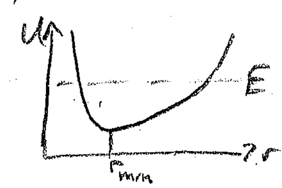
Origin is at center

• Solution in polar coordinates

+ Again, with a central force, motion lies in a plane. Choose xy plane

+ This is motion in effective potential

$$U(r) = \frac{1}{2}kr^2 + \frac{J^2}{2mr^2}$$



+ The radius oscillates between values where $U(r) = E$. If E = minimum value of U, $r = r_{min}$ = constant, and the motion is circular. This is "force balancing."

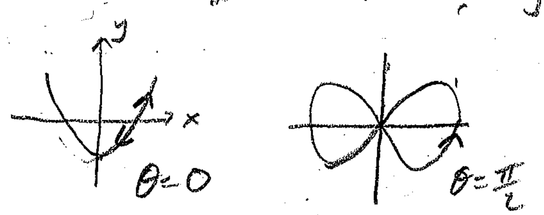
• Anisotropic oscillator: $\omega_x \neq \omega_y \neq \omega_z$

+ Can't rotate to line up max displacement along x

+ Shape of particle path depends on frequency ratio, phase, amplitudes

Ex In 2D: $x = A_x \cos(\omega_x t)$, $y = A_y \cos(\omega_y t - \theta)$

$\omega_y = 2\omega_x$
$A_x = A_y$



etc

+ If ω_y/ω_x = irrational, path in xy plane (etc) does not close!