PHYS-4602 Homework 8 Due 19 March 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Perturbation Theory vs Variational Principle**
   (a) Consider a particle moving in the 1D anharmonic oscillator potential
   $$V(x) = \frac{1}{2}m\omega^2 x^2 + gx^3 \tag{1}$$
   On the previous assignment, you found that the ground state energy remained $E_{gs} = \hbar\omega/2$ through first order in perturbation theory.

   Using the variational method, show that the true ground state energy of this potential is unbounded below (that is, if I give you any real number, demonstrate that the ground state energy is less than that number). We say that this potential is unstable and has no ground state. *Hint*: Think about a simple trial wavefunction that approximates a delta function in position.

   (b) *from Griffiths 7.5* Consider a Hamiltonian $H = H_0 + H_1$, where H_0 is exactly solvable and H_1 is small in some sense. Prove that first-order perturbation theory always overestimates the true ground state energy. That is, show that the ground state energy calculated in first-order perturbation theory is greater than (or equal to) the true ground state energy.

2. **Variational Calculations**
   (a) Consider a particle moving in 1D in a potential $V(x) = \alpha|x|$. Find the best possible upper bound on the ground state energy using a gaussian trial wavefunction.

   (b) Consider a particle moving in the 1D interval $0 \le x < L$ with periodic boundary conditions on the wavefunction $\psi(0) = \psi(L)$. The particle experiences a potential $V(x)$. Show that the ground state energy of this system is less than or equal to the average value of $V(x)$ over the range $0 \le x < L$.

3. **WKB as $\hbar$ Expansion** *Based on Griffiths 8.2*

   In this problem, you'll derive the WKB wavefunction in regions where $E > V(x)$. We will use the 1D Schrödinger equation in the form
   $$\frac{d^2\psi}{dx^2} = -\frac{p(x)^2}{\hbar^2}\psi \ , \quad p(x) = \sqrt{2m(E - V(x))} \ . \tag{2}$$
   (a) Begin by writing the wavefunction as $\psi(x) = \exp[if(x)/\hbar]$ for some complex function f (note that this is completely general). Use the Schrödinger equation (2) to find a second order differential equation for f.

   (b) Now treat $\hbar$ as a small parameter, expanding $f(x) = f_0(x) + \hbar f_1(x) + \cdots = \sum_n \hbar^n f_n(x)$. Then write your differential equation from part (a) as a series in $\hbar$. Find the differential equations that come from the $\hbar^0$, $\hbar^1$, and $\hbar^2$ terms; these should vanish separately as $\hbar \to 0$.

   (c) Find f_0 and f_1 in terms of $p(x)$, and use those to derive the WKB form of the wavefunction. *Hint:* Recall that $\ln z = \ln|z| + i\theta$, where the complex $z = |z|e^{i\theta}$.

   (d) Describe how you would change this procedure for regions where $V(x) > E$.