PHYS-4602 Homework 6 Due 5 March 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Relativistic Harmonic Oscillator**

   Recall that the relativistic energy is $\sqrt{(\vec{p}c)^2 + (mc^2)^2} \approx mc^2 + \vec{p}^2/2m - \vec{p}^4/8m^3c^2$ (plus higher-order corrections), so a 1D harmonic oscillator with the first relativistic correction has the Hamiltonian

   $$H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{1}{2}m\omega^2 x^2 \ . \tag{1}$$

   (a) Find the ground state energy of this oscillator using first-order perturbation theory. What condition must the frequency satisfy for the relativistic correction to be small?

   (b) Find the ground state eigenstate of this oscillator in terms of the unperturbed oscillator eigenstates using first-order perturbation theory.

2. **Stark Effect** *based on Griffiths 6.36*

   The presence of an external electric field $E_0\hat{z}$ shifts the energy levels of a hydrogen atom, which is called the Stark effect. Consider the hydrogen atom to be described by the Coulomb potential; the external electric field introduces a perturbation

   $$H_1 = eE_0 z = eE_0 r\cos\theta \ . \tag{2}$$

   We have already seen on homework that the expectation value of this Hamiltonian in the ground state $n = 1$ vanishes, so there is no shift in the ground state energy. In this problem, we consider the degenerate perturbation theory of the $n = 2$ states. *As spin does not enter, do not consider it in this problem.*

   (a) The four states $|2,0,0\rangle$, $|2,1,0\rangle$, and $|2,1,\pm1\rangle$ are degenerate at 0th order. Label these states sequentially as $i = 1,2,3,4$. Show that the matrix elements $W_{ij} = \langle i|H_1|j\rangle$ form the matrix

   $$W = -3aeE_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \tag{3}$$

   where empty elements are zero and a is the Bohr radius. *Hint*: Note that L_z commutes with H_1, so only states with the same quantum number m can have nonzero matrix elements; this will save you quite a bit of work. Then use the angular wavefunctions to see that all the diagonal elements of W must vanish. Finally, use the explicit wavefunctions to evaluate the remaining matrix elements of W (there should only be one independent one left).

   (b) Diagonalize this matrix to show that $|\pm\rangle = (1/\sqrt{2})(|2,0,0\rangle \pm |2,1,0\rangle)$ are eigenstates of W. Find the first order shift in energies of $|\pm\rangle$. *Hint*: Note that the corrected eigenstates may still have contributions from other values of the principal quantum numbers n, but that doesn't quite matter.

   (c) Finally, show that the states $|\pm\rangle$ have a nonzero dipole moment $p_z = -e\langle z\rangle$ and calculate it. You should not need to do any more calculations; just use your answer from part (b).

3. **Weak-Field Zeeman Effect**

In the class notes, we stated that placing a hydrogen atom in a constant magnetic field $B_0\hat{z}$ introduces a contribution to the hydrogen atom of $H_1 = (e/2m)B_0(L_z + 2S_z)$. If this contribution is larger than the energy level splitting due to fine structure, this gives the "strong-field" Zeeman effect that we discussed in class. In this problem, consider the opposite limit, in which H_1 is smaller than the fine structure splitting. In this case, we include the fine structure corrections in the "unperturbed" Hamiltonian H_0 and treat H_1 as the perturbation to that.

(a) With fine structure included, the eigenstates of H_0 are identified by n, total angular momentum quantum number j, its z component m_j, and the total orbital angular momentum quantum number ℓ (as well as total spin $s = 1/2$); the z-components m_ℓ and m_s are not good quantum numbers. Write $H_1 = (e/2m)B_0(J_z + S_z)$ since $\vec{J} = \vec{L} + \vec{S}$ and show that the change in energy due to B_0 is

$$E^1_{n,j,m_j,\ell} = \frac{e\hbar}{2m} B_0 m_j \left[1 \pm \frac{1}{2\ell + 1}\right] \,. \tag{4}$$

To do this, you will need to know that the eigenstate of J^2, J_z, and L^2 is written

$$\begin{aligned} |j = \ell \pm 1/2, m_j, \ell\rangle \;\; &= \;\; \sqrt{\frac{\ell \mp m_j + 1/2}{2\ell + 1}}|\ell, m_\ell = m_j + 1/2, m_s = -1/2\rangle \\ &\quad \pm\sqrt{\frac{\ell \pm m_j + 1/2}{2\ell + 1}}|\ell, m_\ell = m_j - 1/2, m_s = 1/2\rangle \end{aligned} \tag{5}$$

in terms of the eigenstates of L^2, L_z, and S_z. *Hint:* It may be useful to note that $[H_0, J_z] = [H_1, J_z] = 0$.

(b) The quantity in square brackets in (4) is called the Landé g factor. Show that the g factor can also be written as

$$\left[1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}\right] \,, \tag{6}$$

which is the form given in Griffiths. You can start with (6) and try $j = \ell \pm 1/2$ separately to get the form given in (4).