PHYS-4602 Homework 5 Due 27 Feb 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dirac Notation on the Circle**

   Consider the Hilbert space of L^2 functions on the interval $0 \leq x \leq 2\pi R$ with periodic boundary conditions.

   (a) Show explicitly that the complex exponentials $|e_n\rangle \simeq e^{inx/R}/\sqrt{2\pi R}$ for n any integer form an orthonormal set (as we stated in class). As it turns out, they make a complete orthonormal basis (but you do not have to prove that).

   Carry out the following calculations *without doing any integrals.*

   (b) Calculate the inner product of $|f\rangle \simeq f(x) = \cos^3(x/R)$ and $|g\rangle \simeq g(x) = \sin(3x/R)$.

   (c) Find the inner product of $|f\rangle$ and $|g\rangle$ from part (b) with $|h\rangle \simeq h(x) = \sin(3x/R + \theta)$.

   (d) $|f\rangle, |g\rangle, |h\rangle$ are not normalized. Find their norms.

2. **Gaussian Wavepacket**

   Here we consider the Gaussian wavepacket in 1D at a single instant $t = 0$, ignoring its time evolution. The state is

   $$|\psi\rangle = \int_{-\infty}^{\infty} dx\ Ae^{-ax^2}|x\rangle\ . \tag{1}$$

   Some of these results may be useful on future assignments.

   (a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates. The textbook cover also has a formula for Gaussian integrals.

   (b) Since the wavefunction is even, $\langle x\rangle = 0$. Find $\langle x^2\rangle$. *Hint:* You can get a factor of x^2 next to the Gaussian by differentiating it with respect to the parameter a.

   (c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

   (d) Find $\langle p\rangle$ and $\langle p^2\rangle$ and show that this state saturates the Heisenberg uncertainty principle. You should not have to do any integrations.

3. **Harmonic Oscillator Matrix Elements**

   (a) Calculate the matrix elements $\langle n|x|n'\rangle$ and $\langle n|p^2|n'\rangle$ for $|n\rangle, |n'\rangle$ stationary states of the harmonic oscillator. You *must* use Dirac and operator notation and *may not* carry out any integrals.

   (b) Suppose the system is in the state $|\psi\rangle = (|0\rangle + 2e^{i\theta}|1\rangle)/\sqrt{5}$. Using your previous result, find $\langle x\rangle$ as a function of θ and explain the relation of your answer to the time evolution of a particle initially in that state with $\theta = 0$.

(c) *from a Griffiths problem* A *coherent state* $|\alpha\rangle$ of a harmonic oscillator is an eigenstate of the lowering operator

$$a|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{2}$$

where the eigenvalue α is complex in general ($a \neq a^\dagger$ is not Hermitian). Find the expectation values of x and p in the coherent state $|\alpha\rangle$.

4. **Former Test Question 1**

   Two Hermitian operators A and B have simultaneous eigenstates denoted $|a, b\rangle$, where a is the eigenvalue of A and b is the eigenvalue of B. Answer the following questions as True or False. Explain your reasoning in one sentence per part.

   (a) The state $(|a_1, b\rangle + |a_2, b\rangle)/\sqrt{2}$ with $a_1 \neq a_2$ is an eigenstate of A.

   (b) The state $(|a_1, b\rangle + |a_2, b\rangle)/\sqrt{2}$ with $a_1 \neq a_2$ is an eigenstate of B.

   (c) The vector $(|a_1, b\rangle + 4|a_2, b\rangle)/3$ is correctly normalized.

5. **Former Test Question 2**

   Some operator Q commutes with the Hamiltonian. If the initial state $|\Psi(t = 0)\rangle$ of the system is an eigenstate of Q with eigenvalue q, prove that $|\Psi(t)\rangle$ is also an eigenstate of Q with the same eigenvalue for any time t.

6. **Former Test Question 3**

   Answer the following about 1-qbit and 2-qbit gates.

   (a) Show that acting with $\mathbb{H}$, then $R(\pi)$, then $\mathbb{H}$ again reproduces the NOT gate (meaning NOT is not an independent gate).

   (b) It is not possible to clone a qbit, but is it possible for a unitary gate ($U^\dagger = U^{-1}$) to swap two qbits? That is, does there exist U such that $U(|\psi\rangle_1|\phi\rangle_2) = |\phi\rangle_1|\psi\rangle_2$ for two unknown qbits? *Hint*: choose a basis for the 2-qbit Hilbert space and write U as a matrix.