PHYS-4602 Homework 4 Due 6 Feb 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Entanglement and Gates**

   (a) Start with two qbits in the state $|0\rangle|0\rangle$. Act on the first qbit with the Hadamard operator $\mathbb{H}$ and then use the CNOT operator. What is the total state $|\psi\rangle$?

   (b) Now we will see that qbits 1 and 2 are entangled in state $|\psi\rangle$. Write the density operator $\rho = |\psi\rangle\langle\psi|$ in terms of the 1-qbit states for qbits 1 and 2. Then find the density operator ρ_1 of qbit 1 only by "tracing out" qbit 2 (ie, $\rho_1 = \sum_\alpha {}_2\langle\alpha|\rho|\alpha\rangle_2$ for $\alpha = 0, 1$). Find the eigenvalues of ρ_1. The qbits are entangled unless the two eigenvalues are 0 and 1.

   (c) Act with the Hadamard operator on each qbit of the state $|\psi\rangle$ you found in part (a). That is, find $\mathbb{H}_1\mathbb{H}_2|\psi\rangle$. Is this state entangled?

2. **Square Root of NOT**

   In quantum computing, the NOT gate reverses $|0\rangle$ and $|1\rangle$ qbits. In the standard matrix form, NOT is represented by the Pauli matrix σ_x.

   (a) Show that the operator U given by the matrix

   $$U \simeq \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \tag{1}$$

   is the "square root" of NOT in the sense that $U^2 \simeq \sigma_x$ and also demonstrate that U is unitary ($U^\dagger U = 1$).

   (b) Consider the matrix $e^{-i\theta}\exp(i\theta\sigma_x)$. Show that this is U for $\theta = \pi/4$ and NOT for $\theta = \pi/2$, also demonstrating that U is the square root of NOT.

   (c) Find the operator $e^{i\pi/4}U$ in terms of the phase rotation gate $R(\phi)$ and the Hadamard gate $\mathbb{H}$. You will need to choose a particular value for the phase ϕ.

3. **Cloning Means FTL Communication** *based on a problem by Wilde*

   Suppose that Alice and Bob are at two ends of an EPR/Bell experiment. In other words, they are at rest with respect to each other and separated by 5 lightyears, and each receives one of a pair of entangled electrons with total spin state $s = 0$ simultaneously (in their common rest frame). By prior agreement, Alice measures either the S_z or S_x spin of her electron as soon as she receives it, but Bob does not know which spin she measures.

   After Alice's measurement (in their rest frame time), Bob's electron is in some state $|\psi\rangle_B$. Suppose, in contradiction to the no-cloning theorem, Bob can clone his electron's state onto a large number N of other electrons. (For example, Bob can do some quantum operation that takes his $N+1$ electrons from state $|\psi\rangle_B|\uparrow\rangle_1\cdots|\uparrow\rangle_N$ to state $|\psi\rangle_B|\psi\rangle_1\cdots|\psi\rangle_N$.) What measurement(s) can Bob do on his extra N electrons that will tell him with great certainty whether Alice measured the S_z or S_x spin of her electron? Explain your answer. (Note that Bob can accomplish his measurement before Alice can tell him her measurement choice, so they can establish faster-than-light communication in this way. This is a good reason for the no-cloning theorem!)