PHYS-4602 Homework 3 Due 30 Jan 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Commutators and Functions of Operators**

   (a) Suppose $|a\rangle$ is an eigenfunction of some operator A, $A|a\rangle = a|a\rangle$. Consider the inverse operator A^{-1} defined such that $AA^{-1} = A^{-1}A = 1$. Show that $|a\rangle$ is an eigenvector of A^{-1} with eigenvalue $1/a$ if $a \neq 0$ (if there is an eigenvalue $= 0$, A is not invertible).

   (b) For any function $f(x)$ that can be written as a power series

   $$f(x) = \sum_n f_n x^n \ , \tag{1}$$

   we can define

   $$f(A) = \sum_n f_n A^n \ , \tag{2}$$

   where A^n denotes operating with A n times. Show that

   $$f(A)|a\rangle = f(a)|a\rangle \ . \tag{3}$$

   Does this result hold if the power series includes negative powers?

   (c) For any three operators A, B, C, show that

   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{4}$$

   (d) Then prove by induction that

   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{5}$$

   if $[A, B]$ commutes with B (for $n > 0$).

   (e) Finally, show using (5) that $[p, f(x)] = -i\hbar df/dx$, where x and p are 1D position and momentum operators with $[p, x] = -i\hbar$. Assume $f(x)$ can be written as a Taylor series.

2. **Measurement vs Time Evolution** *a considerable expansion of Griffiths 3.27*

   Suppose a system has observable A with eigenstates $|a_1\rangle, |a_2\rangle$ of eigenvalues a_1, a_2 respectively and Hamiltonian H with eigenstates $|E_1\rangle, |E_2\rangle$ of energies E_1, E_2 respectively. The eigenstates are related by

   $$|a_1\rangle = \frac{1}{5}\left(3|E_1\rangle + 4|E_2\rangle\right) \ , \quad |a_2\rangle = \frac{1}{5}\left(4|E_1\rangle - 3|E_2\rangle\right) \ . \tag{6}$$

   Suppose the system is measured to have value a_1 for A initially. Each of the following parts asks about a different possible set of subsequent measurements.

   (a) What is the probability of measuring energy E_1 immediately after the first measurement? Assuming we do get E_1, what is the probability of measuring a_1 again if we measure A again immediately after the measurement of energy?

   (b) Instead, consider immediately measuring A again after the first measurement. What are the probabilities for observing a_1 and a_2?

(c) Finally, consider making the first measurement and then allowing the system to evolve for time t. If we then measure energy, what is the probability of finding energy E_1? If we instead measured A again, what is the probability we find a_1 again?

3. **Matrix Hamiltonian**

Consider a 3D Hilbert space with Hamiltonian

$$H \simeq E_0 \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{7}$$

in some basis. Work in this basis throughout the problem.

(a) Show that the time evolution operator is

$$e^{-iHt/\hbar} \simeq \cos(E_0 t/\hbar) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - i \sin(E_0 t/\hbar) \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{8}$$

in this basis.

(b) Some operator A is defined in this basis as

$$A \simeq A_0 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \tag{9}$$

Suppose the system starts out at time $t = 0$ in a state represented by $[1\ 0\ 0]^T$. Using your previous result, find the state of the system and $\langle A \rangle$ as a function of time. At what times is $\langle A \rangle$ minimized?