PHYS-4602 Homework 1 Due 16 Jan 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dual Vectors and Change of Basis**

   Consider a 3-dimensional complex column vector space, which has the usual orthonormal basis

$$|e_1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \ |e_2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \ |e_3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ . \tag{1}$$

   In that basis, the vectors $|f_i\rangle$ $(i = 1, 2, 3)$ can be written as

$$|f_1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \ |f_2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad , \ |f_3\rangle \simeq \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ -i \\ -2i \end{bmatrix} \ . \tag{2}$$

   (a) Write the $|f_i\rangle$ as linear superpositions of the $|e_i\rangle$ basis vectors.

   (b) Show that the $|f_i\rangle$ are normalized and mutually orthogonal, so they form a complete orthonormal basis (distinct from the set of $|e_i\rangle$).

   (c) Write the associated dual vectors $\langle f_i|$ as row vectors in the $\{\langle e_i|\}$ basis.

   (d) Write the $|e_i\rangle$ vectors as linear superpositions of the $|f_i\rangle$. Use your result to do a change of basis for this Hilbert space by writing the $|e_i\rangle$ vectors as column vectors in the $\{|f_i\rangle\}$ basis. *Hint:* You can solve a system of linear equations or use a similarity transformation, but it is much easier if you use inner products as discussed in the notes.

2. **Superposition of States**

   Suppose $|\psi\rangle$ and $|\phi\rangle$ are two normalized state vectors, and so is $|\alpha\rangle = A(3|\psi\rangle + 4|\phi\rangle)$.

   (a) Find the normalization constant A in the case that

      i. $\langle\psi|\phi\rangle = 0$.

      ii. $\langle\psi|\phi\rangle = i$.

      iii. $\langle\psi|\phi\rangle = e^{i\pi/6}$.

   (b) In the case $\langle\psi|\phi\rangle = i$, find the part of $|\alpha\rangle$ orthogonal to $|\psi\rangle$. Verify that it is orthogonal by taking the inner product. You may use the *Gram-Schmidt procedure* described in Griffiths problem A.4 to

   (c) Now suppose that $\langle\psi|\phi\rangle = 0$ and define a new state $|\beta\rangle = B(4e^{-i\theta}|\psi\rangle + 3e^{i\theta}|\phi\rangle)$ for some angle θ. Find the normalization constant B and $\langle\alpha|\beta\rangle$ (you make assume that the normalization constants are positive and real).

3. **1-Qbit Density Matrix** *inspiration from Griffiths & Schroeter 12.6 & 12.8*

   Consider the density matrix ρ for a single qbit (you may consider this to be the spin of a single spin-1/2 particle instead). In this problem, describe ρ as a matrix rather than an abstract operator.

   (a) Prove that $\rho^2 = \rho$ if and only if the state is pure. *Hint:* Think about the diagonal form of ρ in pure and mixed states.

(b) Using the requirements that $\mathrm{Tr}(\rho) = 1$ and $\rho^\dagger = \rho$, show that the most general density matrix for a single qbit is

$$\rho = \frac{1}{2} \begin{bmatrix} (1 + a_3) & (a_1 - ia_2) \\ (a_1 + ia_2) & (1 - a_3) \end{bmatrix} , \tag{3}$$

where $a_{1,2,3}$ are real numbers. (This can also be written in terms of the Pauli sigma matrices as $(1 + \vec{a} \cdot \vec{\sigma})/2$.)

(c) Define the *Bloch vector* $\vec{a}$ as the vector with components $a_{1,2,3}$. Use part (a) to show that ρ represents a pure state if $|\vec{a}| = 1$ and a mixed state if $|\vec{a}| < 1$ (that is, the Bloch vector lies on the surface of the Bloch sphere for a pure state and inside the Bloch sphere for a mixed state).

(d) Find the quantum von Neumann entropy for $a_1 = 1/2$, $a_2 = 0$, and $a_3 = 0$. *Hint:* you may want to diagonalize ρ first.