Time Dependence

- Schrödinger Equation, Axiom

In QM, a time-dependent state $|\Psi(t)\rangle$ evolves according to the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

where H is the Hamiltonian operator (total energy in many contexts), which is Hermitian

- Stationary States
  + In general, the Schr. eqn. can be thought of as a vector of differential equations (or PDE)
  + But consider writing $|\Psi(t)\rangle$ in the eigenbasis of H (energy eigenbasis) $\{|E_n\rangle\}$

$$|\Psi(t)\rangle = \sum_n c_n(t)|E_n\rangle$$

  + Then take $\langle E_n|$ on the Schr. eqn to get

$$\dot{c}_n = (-iE_n/\hbar)c_n$$
$$\Rightarrow \quad c_n = c_n^0 \exp(-iE_nt/\hbar)$$

If we know energy eigenstates, we know time evolution!

  + If $|\Psi(t=0)\rangle = |E_n\rangle$ is an energy eigenstate,

$$|\Psi(t)\rangle = e^{-iE_nt/\hbar}|E_n\rangle$$

This means that all probabilities $P_a = |\langle a|\Psi(t)\rangle|^2 = |\langle a|E_n\rangle|^2$ and expectation values $\langle A\rangle = \langle E_n|A|E_n\rangle$ are time-indep. Energy e'states are called stationary states

- Ehrenfest's Theorem: expectation values obey classical physics
  + Consider some observable $A(t)$ w/ explicit time dependence & its expectation value $\langle A\rangle(t) = \langle\Psi(t)|A(t)|\Psi(t)\rangle$
  + Schr. eqn is

$$\frac{d}{dt}|\Psi(t)\rangle = \frac{-i}{\hbar}H|\Psi(t)\rangle \quad \text{and} \quad \frac{d}{dt}\langle\Psi(t)| = \frac{+i}{\hbar}\langle\Psi(t)|H$$

+ Therefore,

$$d\langle A\rangle/dt = \frac{i}{\hbar}\langle\Psi|HA|\Psi\rangle + \langle\Psi|\frac{\partial A}{\partial t}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|AH|\Psi\rangle$$

$$= \frac{i}{\hbar}\langle[H,A]\rangle + \langle\frac{\partial A}{\partial t}\rangle$$

This is <u>Ehrenfest's</u> <u>Theorem</u>

+ Consider position x and momentum p operators with $[x,p]=i\hbar$ and $H = p^2/2m + V(x)$. Then

$$[H,x] = [p^2,x]/2m = \frac{-i\hbar}{m}p,\quad [H,p] = -[p,V(x)] = i\hbar\frac{dV}{dx}$$

so

$$d\langle x\rangle/dt = \langle p\rangle/m \quad\text{and}\quad d\langle p\rangle/dt = i\hbar\langle dV/dx\rangle$$

as expected classically.

+ There is a more general connection to the Hamiltonian version of classical mechanics.

— Example: Neutrino Oscillation (2015 Nobel Physics)

- Flavor vs Mass

  + Subatomic "matter" particles come in 3 <u>families</u>, which are copies of each other except for different masses. Corresponding particles in different families have different <u>flavor</u>

  + The electron & the (neutral) <u>electron neutrino</u> are in 1st family. The other families are "muon flavor" and "tau flavor"

    $m_e < m_\mu < m_\tau$, m_ν's = ?

e	ν_e
μ	ν_μ
τ	ν_τ

  + Standard Model interactions create neutrinos of a definite flavor.

- What's the Hamiltonian?

  + Consists of kinetic H_K and interaction H_I parts

  + Flavor states are e'states of H_I (ν, other stuff), but not H_K.

  + While neutrinos travel, they aren't interacting, so $H = H_K$, and we need to know the H_K or "mass" eigenstates

  + We define $|\nu_a\rangle = \sum_i U_a{}^i|\nu_i\rangle$ $a=1,2,3$ mass, $i=e,\mu,\tau$ flavors

- Kinetic Hamiltonian: Simplified 2 neutrino model
  + Need a relativistic Hamiltonian/energy.
    Remember special relativity $H_K = \sqrt{(\vec{p}c)^2 + (mc^2)^2}$
  + Normally, we have a nonrelativistic approx
    $$H_K \approx mc^2 + \vec{p}^2/2m + \cdots$$
  + But for very energetic particles $\vec{p}^2 \gg m^2c^2$
    $$H_K \approx |\vec{p}|c + m^2c^3/2|\vec{p}| + \cdots$$
    This is our case.
  + For a neutrino, $\vec{p}$ is conserved, but there are multiple mass states. In a 2 neutrino model,
    $$H_K = |\vec{p}|c\,\mathbb{1} + \frac{c^3}{2|\vec{p}|}\begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix} \quad \text{in } \underline{\text{mass basis}}$$

- Suppose our initial state is a flavor state, say $|\nu_e\rangle$.
  + In the mass basis, our 2 flavor states are
    $$|\nu_e\rangle \simeq \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},\ |\nu_\mu\rangle \simeq \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},\ \theta = \text{mixing angle}$$
  + The time evolution is
    $$|\Psi(t)\rangle = e^{-ipct/\hbar}\begin{bmatrix} \cos\theta\, \exp(-ic^3m_1^2t/2p\hbar) \\ \sin\theta\, \exp(-ic^3m_2^2t/2p\hbar) \end{bmatrix}$$
  + After time t, probability of interacting with/measuring $|\nu_\mu\rangle$ is
    $$P_\mu = |\langle \nu_\mu | \Psi(t)\rangle|^2 = \left| [-\sin\theta\ \cos\theta] \begin{bmatrix} \cos\theta \exp(\cdots) \\ \sin\theta \exp(\cdots) \end{bmatrix} \right|^2$$
    $$= \sin^2\theta\cos^2\theta \left| e^{-ic^3m_1^2t/2p\hbar} - e^{-ic^3m_2^2t/2p\hbar} \right|^2$$
    $$= \sin^2(2\theta)\, \sin^2\left[\frac{(m_2^2 - m_1^2)c^3t}{4p\hbar}\right]$$

  + Neutrinos move at close to c, so $pc \simeq E$, $ct \simeq L$ (distance), so famous formula is
    $$P_\mu = \sin^2(2\theta)\, \sin^2\left[\Delta m^2 c^3 L/4E\hbar\right]$$

  + Relation of flavor to mass basis more complicated w/ 3 neutrinos + includes complex phase.

- Oscillation of probabilities + expectation values is general behavior when the state is a superposition of stationary states. Happens elsewhere in particle physics, too

— Time Evolution Operator

- We can reformulate our solution to Schr. eqn. as follows
  + Write out state in the energy basis as

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar}|E_n\rangle = \sum_n c_n e^{-iHt/\hbar}|E_n\rangle$$

$$= e^{-iHt/\hbar}\left(\sum_n c_n|E_n\rangle\right) = e^{-iHt/\hbar}|\Psi(0)\rangle$$

  + Note that

$$i\hbar\frac{d}{dt}\left(e^{-iHt/\hbar}|\Psi(0)\rangle\right) = H\left(e^{-iHt/\hbar}|\Psi(0)\rangle\right)$$

  agrees with Schr. eqn

  + So we define $U(t) = \exp(-iHt/\hbar)$ as the time-evolution operator

- The time evolution operator is unitary
  + Expand it out:

$$U^\dagger U = \left(1 + iHt/\hbar - \frac{1}{2}\left(\frac{Ht}{\hbar}\right)^2 + \dots\right)\left(1 - \frac{iHt}{\hbar} - \frac{1}{2}\left(\frac{Ht}{\hbar}\right)^2 \dots\right)$$

$$= 1 + \left(\frac{iHt}{\hbar} - \frac{iHt}{\hbar}\right) + \left[(Ht/\hbar)^2 - \frac{2}{2}(Ht/\hbar)^2\right] + \dots = 1$$

  + Any operator (like an observable) takes a state to a vector. Time evolution (or any process including a change of reference frame, ~~sef~~ axis rotation) must take a state to a state (normalized)
  + So if $|\psi'\rangle = U|\psi\rangle$ for some operator U, we need

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle \overset{?}{=} \langle\psi|\psi\rangle = 1$$

  This is only true for every state $|\psi\rangle$ if $U^\dagger U = 1$ or U is unitary.

  + This is an important basic feature of QM due to linear algebra

- Alternate Pictures
  + We have so far treated time evolution this way:
    1) Operators are fixed, so eigenvectors / eigenbasis for each observable are fixed
    2) The state $|\Psi(t)\rangle$ of a system evolves according to Schr. eqn via $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$

  Called Schrödinger picture

+ We could instead

1) Keep the physical state $|\Psi\rangle$ time independent

2) Have observable operators time evolve as

$$A(t) = U^\dagger(t) A(0) U(t)$$

Expectation values are same as above. This is Heisenberg picture

+ Particle physics commonly uses something in between

- Energy-Time Uncertainty "Principle"

Much more heuristic than the real one

- As usual, define $\Delta E = \sigma_H$. What's Δt?

+ For any observable A,

$$\sigma_H \sigma_A \geq \frac{1}{2} |\langle [H, A] \rangle|$$

+ Assuming no explicit time dependence,

$$|\langle [H, A] \rangle| = \hbar \, |d\langle A \rangle / dt|$$

- What's Δt? + Define Δt to be the time it takes for our operator to change significantly.

$$\sigma_A \sim \Delta t \, |d\langle A \rangle / dt|$$

+ Altogether,

$$\Delta E \, \Delta t \gtrsim \hbar/2$$

+ Can use this for back-of-envelope calculations.