- Measurements + Probability

- Observables Quantities, Axiom:
An observable is a Hermitian linear operator on the Hilbert space of states, and any measurement of that observable yields an eigenvalue of that operator

- Operators: turns a vector into another vector
  + A linear operator obeys superposition
$$A(|\psi\rangle + |\phi\rangle) = A|\psi\rangle + A|\phi\rangle$$
  + Its action is given by a matrix for a given basis
Define
$$A|e_i\rangle = \sum_j A_{ji}|e_j\rangle$$
Then for
$$|\psi\rangle = \sum c_i|e_i\rangle \simeq \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}$$
we see
$$A|\psi\rangle \simeq \begin{bmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}$$
  + For an orthonormal basis the matrix elements are therefore $A_{ij} = \langle e_i|A|e_j\rangle$ (inner product)
We call a general inner product $\langle\phi|A|\psi\rangle$ a matrix element

- Hermitian Adjoint
  + The (Hermitian) adjoint of operator A that is, like A "acting to the left" in an inner product.
We define $A^\dagger$ such that
$$\langle\phi|A^\dagger|\psi\rangle = (\langle\psi|A|\phi\rangle)^*$$
in terms of the complex conjugate + inner product
  + In matrix form, let
$$|\psi\rangle = \psi_1|e_1\rangle + \cdots + \psi_N|e_N\rangle, \quad |\phi\rangle = \phi_1|e_1\rangle + \cdots + \phi_N|e_N\rangle$$
so
$$\langle\phi|A|\psi\rangle = [\phi_1^* \cdots \phi_N^*]\begin{bmatrix} A_{ij} \end{bmatrix}\begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} = [\psi_1 \cdots \psi_N]\begin{bmatrix} A^T \end{bmatrix}\begin{bmatrix} \phi_1^* \\ \vdots \\ \phi_N^* \end{bmatrix}$$
$$= \left([\psi_1^* \cdots \psi_N^*]\begin{bmatrix} A^* \end{bmatrix}\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_N \end{bmatrix}\right)^* \Rightarrow A^\dagger = (A^T)^* \text{ for matrices}$$

+ A Hermitian operator satisfies $A = A^\dagger$. A unitary operator satisfies $A^\dagger = A^{-1}$

- Eigenvalues + Eigenvectors
  + An eigenvector of operator A is a non-zero vector $|\lambda\rangle$ such that $A|\lambda\rangle = a|\lambda\rangle$, where a is a scalar called the eigenvalue (a can be $= 0$).
    Often an eigenvector is labeled by the eigenvalue $|\lambda\rangle \equiv |a\rangle$, so a is both a number + a label
  + We can write the eigenvector eqn with the help of the identity operator as $(A - a\mathbb{1})|a\rangle = 0$.
    When we write operators as matrices,
    $$\Rightarrow \det(A - a\mathbb{1}) = 0$$
    In finite dimensions, this characteristic equation is for roots of a polynomial
  + For ~~eit~~ Hermitian operators: eigenvalues are real, eigenvectors can be made into an orthonormal basis.
    Partial proof: Consider 2 eigenvectors $|a_1\rangle, |a_2\rangle$ of A
    $$\langle a_2|A|a_1\rangle = a_1\langle a_2|a_1\rangle$$
    $$\text{also} = (\langle a_1|A^\dagger|a_2\rangle)^* = (\langle a_1|A|a_2\rangle)^* = a_2^*\langle a_2|a_1\rangle$$
    If $|a_1\rangle = |a_2\rangle$, we find $a_1 = a_1^* = $ real.
    If $|a_1\rangle \neq |a_2\rangle$, they are orthogonal if $a_1 \neq a_2$
    (If e'values are equal, use Gram-Schmidt)
  + A+B are compatible if the eigenbasis of A can be written as an eigenbasis of B.
    This happens iff $[A,B] = AB - BA = 0$
- Dyads
  + A dyad is an operator written as a sum of terms like $|\phi\rangle\langle\psi|$. The density operator is an example
    If we think about this in terms of matrices, this is column x row = square matrix.
  + The identity operator is
    $$\mathbb{1} = \sum_i |e_i\rangle\langle e_i| \quad \text{a whole orthonormal basis}$$
    This is called the completeness relation
  + Any operator in a given basis is
    $$A = \sum_{ij} A_{ij} |e_i\rangle\langle e_j|$$

A Hermitian operator in its own eigenbasis is

$$A = \sum_n a_n |a_n\rangle\langle a_n|$$

b/c $\langle a_n|A|a_m\rangle = a_m \langle a_n|a_m\rangle = a_m \delta_{mn} = A_{nm}$

- Measurement Axiom:

If the state of the system is $|\psi\rangle$, a measurement of A returns eigenvalue a_n with probability $P_n = |\langle a_n|\psi\rangle|^2$

- Probability is often defined as the frequency of a given result in many experimental trials
  - This is not repeated measurements of 1 system in QM
  - It is measurements on a large ensemble of identically prepared systems
  - The rule for probabilities means they add to 1 (good!)

$$\sum_n P_n = \sum_n \langle\psi|a_n\rangle\langle a_n|\psi\rangle = \langle\psi|\psi\rangle = 1$$

by the completeness relation

- Without measuring the whole probability distribution, we can look at its moments
  - The expectation value (or mean) of an observable in repeated experiments is

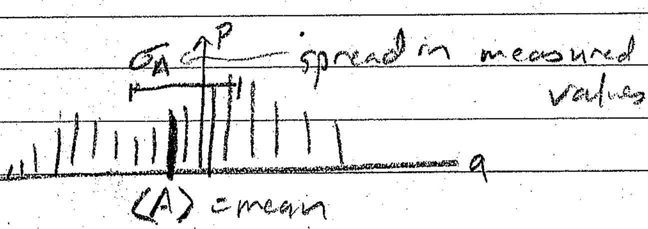
$$\langle A\rangle = \sum_n a_n P_n = \langle\psi|\Big(\sum_n a_n |a_n\rangle\langle a_n|\Big)|\psi\rangle = \langle\psi|A|\psi\rangle$$

The expectation value is a "diagonal matrix element"

  - The width of the distribution is the uncertainty (standard deviation) $\sigma_A = \langle(\Delta A)^2\rangle$ with $\Delta A = A - \langle A\rangle$
    It's usually easiest to calculate via

$$\langle(\Delta A)^2\rangle = \langle A^2\rangle - 2\langle A\langle A\rangle\rangle + \langle A\rangle^2 = \langle A^2\rangle - \langle A\rangle^2$$

  - Graphically,



• Uncertainty Principle

+ If the system is in any state $|\psi\rangle$, consider the uncertainties of two observables $A + B$. with $|f\rangle = \Delta A|\psi\rangle$, $|g\rangle = \Delta B|\psi\rangle$,
$\sigma_A^2 = \langle f|f\rangle$, $\sigma_B^2 = \langle g|g\rangle$
+ The Schwarz inequality tells us
$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2 \geq (\mathrm{Im}\langle f|g\rangle)^2$$
+ That is
$$\sigma_A^2 \sigma_B^2 \geq -\tfrac{1}{4}\left(\langle f|g\rangle - \langle g|f\rangle\right)^2 = -\tfrac{1}{4}\left(\langle\psi|(\Delta A\Delta B - \Delta B\Delta A)|\psi\rangle\right)^2$$
+ Since $[\Delta A, \Delta B] = [A,B]$ (identity commutes w/everything), we find
$$\sigma_A^2 \sigma_B^2 \geq -\tfrac{1}{4}\langle\psi|[A,B]|\psi\rangle^2$$
By properties of commutators and adjoints,
$$[A,B]^\dagger = [B^\dagger, A^\dagger] = -[A,B] \text{ for Hermitian } A+B$$
$\Rightarrow$ eigenvalues + expectation values are pure imaginary.

- The Measurement Problem + Interpretations of QM
  • We know a measurement of observable A in state $|\psi\rangle$ returns an eigenvalue a_n w/ some probability.
  + At that instant, it seems to us that the state becomes $|a_n\rangle$. This is called <u>collapse of the wavefunction</u> and it is <u>not</u> normal time evolution
  + If there are multiple e'states w/ the same (measured) e'value, the collapse is to $|\psi\rangle \to \sum_i |a_i\rangle\langle a_i|\psi\rangle$ (summed over just those $|a_i\rangle$)

  • The collapse of the wavefunction is problematic. Consider our entangled Schrödinger Cat before measurement
$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\left(|\ \ \rangle|N\rangle + (\text{phase})|\ \ \rangle|N, N_2 \cdots\rangle\right)$$
  + The usual 'wavefunction collapse' is the <u>Copenhagen Interpretation</u>.
  + But does measurement require a conscious observer? (Wigner, Wheeler)
  + Do we need a better definition of measurement?
  + We will generally take this point of view pedagogically for concreteness
"Shut up and calculate!" (Mermin)

- Why isn't the observer/measurement device also quantum? Then nothing should distinguish measurement from normal time evolution via Schrödinger eqn
  + The <u>Many Worlds Interpretation</u> (Everett) says the state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|😺\rangle|N\rangle + (\text{phase})|😿\rangle|N_1 N_2 \cdots\rangle\right)|\text{obs}\rangle|\text{universe}\rangle$$

and evolves to

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|😺\rangle|N\rangle|\text{sees alive}\rangle + (\text{phase})|😿\rangle|N_1 N_2 \cdots\rangle|\text{sees dead}\rangle\right)|\text{univ.}\rangle$$

  + The effect is like wavefunction collapse b/c the observer always sees a definite value, but what happens is the wavefunction branches. Everything possible happens in some branch (the "many worlds")
  + Everything happens by usual Schr. eqn

- Decoherence: The rough idea is that interactions with the environment — or among particles in a large system — forces the state into a definite classical-like eigenstate quickly

– EPR Paradox + Bell's Inequality
- Einstein, Podolsky, Rosen thought experiment
  + Produce $e^{\pm}$ pair in $|s=0\rangle = (|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)/\sqrt{2}$ spin state. Send e^- w/ A 3 lyr from earth, and e^+ w/ B 3 lyr from earth other direction.
  + A+B each measure S_z when they see the same light flash from earth. If A measures $S_z = +\hbar/2$, B must find $-\hbar/2$ + vice versa
- EPR notes that the measurements are spacelike separated, + so information cannot travel between $e^{\pm}$ in time to tell each other how to "collapse wavefunctions"
  + Postulated a "<u>hidden variable</u>" that determines each spin from the start (contradicts usual QM). This means there is a function $A(\hat{a},\lambda) = \pm 1$ that knows in advance whether spin in the $\hat{a}$ direction is $\pm$ for however A might measure (and same for $B(\hat{b},\lambda)$)

- In 1964, J. Bell proposed a modified EPR expt
  - It's the same as EPR except A + B measure different spin components $\hat{a}\cdot\vec{S}_A$ and $\hat{b}\cdot\vec{S}_B$ w/o knowing what each other measure
  - Afterwards, they compare results (from many pairs). "On average", QM predicts

$$P(\hat{a},\hat{b}) \equiv \frac{4}{\hbar}\langle(\hat{a}\cdot\vec{S}_A)(\hat{b}\cdot\vec{S}_B)\rangle = -\hat{a}\cdot\hat{b}$$

This is always -1 when $\hat{a}=\hat{b}$

- Bell also proved an inequality for hidden variables
  - From angular momentum conservation, the 2 hidden variable functions satisfy $A(\hat{a},\lambda) = -B(\hat{a},\lambda)$
  - Given some probability distribution $p(\lambda)$ for the hidden variable, possibly integral

$$P(\hat{a},\hat{b}) = \sum_\lambda p(\lambda) A(\hat{a},\lambda) B(\hat{b},\lambda)$$

$$\Rightarrow P(\hat{a},\hat{b}) - P(\hat{a},\hat{c}) = -\sum_\lambda p(\lambda) A(\hat{a},\lambda)\left[A(\hat{b},\lambda) - A(\hat{c},\lambda)\right]$$

  - Then triangle inequalities and the fact that $|A(\hat{a},\lambda)| = 1$ give

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{c})| \le \sum_\lambda p(\lambda)\,\Big|\underbrace{1 - A(\hat{b},\lambda)A(\hat{c},\lambda)}_{\text{but } \le 1}\Big|$$

$$= 1 + P(\hat{b},\hat{c})$$

- Result + Meaning:
  - The Quantum formula violates Bell's inequality when $\hat{c}$ is "between" $\hat{a}$ + $\hat{b}$ and $\hat{a}\perp\hat{b}$.
  - Experiments verify QM prediction
  - This looks nonlocal (ie, FTL), but you can't send information this way. So QM is causal
  - Instead, say QM is "not real" meaning the $e^{\pm}$ do not have a "real" or definite spin until measured.