Foundations of QM

We will discuss axioms, interpretations, + consequences

Quantum States

- Axiom: States are vectors in a complex vector space that has an inner product (with some conditions)
  This type of vector space is called a Hilbert space.
  - Write a vector as a "ket" $|\psi\rangle$, $|\phi\rangle$, etc
    + You can add vectors together $|\psi\rangle + |\phi\rangle$
    + You can multiply vectors by complex numbers (scalars)

  - Basis vectors:
    + In an N-dimensional space, this is a set of N vectors $\{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}$ such that any vector is a linear superposition of basis vectors

$$|\psi\rangle = c_1 |e_1\rangle + c_2 |e_2\rangle + \cdots + c_N |e_N\rangle, \quad c_i = \text{scalars}$$

    + There are different possible basis sets for any vector space. So for different basis vectors $\{|f_i\rangle\}$,

$$|\psi\rangle = a_1 |f_1\rangle + a_2 |f_2\rangle + \cdots + a_N |f_N\rangle$$

    is the same vector $|\psi\rangle$ (just the representation differs)
    + Any vector space can be written in terms of column matrices by assigning basis vectors

$$|e_1\rangle \doteq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad |e_2\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \text{ etc}$$

    Then

$$|\psi\rangle \doteq \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix} \text{ as above.}$$

    If you use a different basis, you get different components in the columns. (In lin. alg., you learn about this "change of basis" in terms of matrix multiplication)

- Inner Product: Function of 2 vectors that gives a scalar

  + We will write the inner product of $|\phi\rangle$ and $|\psi\rangle$ as $\langle\phi|\psi\rangle$. In our complex Hilbert space, this is $= \langle\psi|\phi\rangle^*$ (complex conjugate)

  + The inner product is linear in right-hand argument and "anti-linear" in the left-hand argument:
  If $|\phi\rangle = a|\alpha\rangle + b|\beta\rangle$ and $|\psi\rangle = c|\gamma\rangle + d|\delta\rangle$,
  $$\langle\phi|\psi\rangle = c\langle\phi|\gamma\rangle + d\langle\phi|\delta\rangle$$
  $$= a^*\langle\alpha|\psi\rangle + b^*\langle\beta|\psi\rangle$$
  $$= a^*c\,\langle\alpha|\gamma\rangle + b^*c\langle\beta|\gamma\rangle + a^*d\,\langle\alpha|\delta\rangle + b^*d\langle\beta|\delta\rangle$$

  + The inner product of a vector with itself is positive real. The norm of $|\psi\rangle$ is $\sqrt{\langle\psi|\psi\rangle}$. A vector $|\psi\rangle$ is n[illegible]

  + When we choose basis vectors, we can choose an orthonormal basis (in finite # of dimensions) meaning $\langle e_i|e_j\rangle = \delta_{ij}$. So $|\psi\rangle = c_1|e_1\rangle + \cdots + c_N|e_N\rangle$ and $|\phi\rangle = b_1|e_1\rangle + \cdots + b_N|e_N\rangle$ have $\langle\phi|\psi\rangle = b_1^*c_1 + \cdots + b_N^*c_N$.
  + With an orthonormal basis
  $$|\psi\rangle = (\langle e_1|\psi\rangle)|e_1\rangle + \cdots (\langle e_N|\psi\rangle)|e_N\rangle$$
  This gives a handy way to find components / change bases

  + The left-hand argument of the inner product is written as a bra $\langle\psi|$. Mathematically, it is called a dual vector, which means it is an instruction to take the inner product of $|\psi\rangle$ with any vector to the right.

  + In terms of matrices, a bra is a row given by the Hermitian adjoint (transpose + conjugate) of the ket
  $$\langle\psi| \simeq [c_1^*\ c_2^* \cdots c_N^*] \simeq (|\psi\rangle)^\dagger$$

- There are other conditions on physical states:

  + A physical state has unit norm $\langle\psi|\psi\rangle = 1$
    Therefore, if $|\psi\rangle = \sum_i c_i |e_i\rangle$, then $\sum_i |c_i|^2 = 1$
    (for orthonormal basis)

  + We will see that $e^{i\theta}|\psi\rangle$ is physically equivalent to $|\psi\rangle$ (where θ is a real angle).
    In other words, changing by an overall complex phase keeps the state the same.

— Important examples / Generalizations

- Spin (+ Angular Momentum) States:

  + Recall that a state of definite spin has 2 quantum numbers s and m.
  + s can be integer or half-integer (non-negative) and $-s \leq m \leq s$ w/ values of m separated by integers

  + So $s=1$ has 3 states with $m=-1,0,1$
    $|1,-1\rangle$, $|1,0\rangle$, $|1,1\rangle$. Commonly assigned
    $$|1,1\rangle \doteq \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad |1,0\rangle \doteq \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad |1,-1\rangle \doteq \begin{bmatrix}0\\0\\1\end{bmatrix}$$

  + $s=1/2$ has 2 states, $m=\pm 1/2$, $|\tfrac{1}{2},\tfrac{1}{2}\rangle \equiv |\uparrow\rangle$, $|\tfrac{1}{2},-\tfrac{1}{2}\rangle \equiv |\downarrow\rangle$
    with
    $$|\uparrow\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix}, \quad |\downarrow\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$$

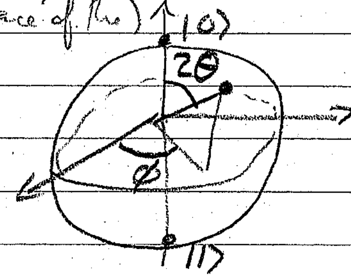
- Qubit (or qbit) states

  + Any 2D Hilbert space, so simplest nontrivial example.
    A spin $s=1/2$ system is a physical representation of a single qbit

+ The typical basis is usually written as $|0\rangle$ and $|1\rangle$ (orthonormal)

+ The general state of 1 qbit is
$$|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\, e^{i\phi}\,|1\rangle$$
Note: (1) normalized (2) we choose overall phase so coefficient of $|0\rangle$ is real, positive (so $0 \le \theta \le \pi/2$)

+ This qbit is a point on the (surface of the) Bloch sphere. At $\theta = 0$, ϕ clearly doesn't matter. At $\theta = \pi/2$, ϕ becomes overall phase + also doesn't matter — these are poles.



• Multiple particles, etc

+ Suppose you have multiple particles, one particle w/ spin + orbital angular momentum, etc.

+ States of a system with multiple parts are from a tensor product vector space. In this case, we can choose a basis where each part corresponds to a factor in the state (factorized basis)

multiparticle state → $|\psi\rangle = |\alpha\rangle_1 |\beta\rangle_2 \cdots$ ← single particle state

+ For example, 2 qubits (with each in the usual basis) has a factorized basis
$$|0\rangle_1|0\rangle_2 \,,\quad |0\rangle_1|1\rangle_2 \,,\quad |1\rangle_1|0\rangle_2 \,,\quad |1\rangle_1|1\rangle_2$$

+ A non-factorized basis is often useful. For example, if there are 2 spins, a factorized state is $|s_1, m_1\rangle|s_2, m_2\rangle$, but a total angular momentum basis (think addition of angular momentum) $|s, m; s_1, s_2\rangle$ can be helpful.

+ Note/Remember: Identical particles can be bosons (if they have integer spin) or fermions (odd half-integer spin)

The state of multiple identical bosons is always <u>symmetric</u> (remains the same) if any 2 single-particle states switch; the state of multiple identical fermions is <u>anti-symmetric</u> (gains a − sign) when 2 single-particle states switch. (These generally don't factorize)

2 bosons : $\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\alpha\rangle_2\right)$ $\quad \left(\langle\alpha|\beta\rangle = 0\right)$

2 fermions : $\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_1|\alpha\rangle_2\right)$

- Consequences
  - Superposition: + The state of a quantum system can be a linear superposition of 2 or more basis states
    + If the assignment of basis has a physical meaning, then a general state may have ambiguous physical properties of some type.
    + This leads to the probabilistic nature of quantum mechanics (we'll return to this later)
  - Entanglement:
    + This is a consequence of superposition for multiparticle systems (tensor product states)
    + Entanglement is when the full multiparticle state does <u>not</u> factorize
    + That means the state of a single particle is not well-defined on its own, it depends on the state of the other particle
    + Examples: Consider the total spin states of an electron + positron (<u>not identical</u>)

$$|s=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right) \quad \text{entangled}$$

$$|s=1, m=1\rangle = |\uparrow\rangle_1|\uparrow\rangle_2 \quad \text{not}$$

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right) \quad \text{entangled}$$

$$|s=1, m=-1\rangle = |\downarrow\rangle|\downarrow\rangle \quad \text{not}$$

Works for qubits, $|\uparrow\rangle \to |0\rangle$, $|\downarrow\rangle \to |1\rangle$

+ the most famous entangled state is Schrödinger's cat: cat in a box with poison controlled by a single radioactive nucleus. After 1 half-life,

$$|\psi\rangle \to \frac{1}{\sqrt{2}}\left(|\text{🐱}\rangle|N\rangle + (\text{phase})\,|\text{😵}\rangle|N_1, N_2 \dots\rangle\right)$$

We'll also come back to this

- Quantum Information (very brief introduction): what is the information content of a state?

- Information is "what we can learn" (Claude Shannon)
  + If we're flipping a fair coin, we don't get much information per flip — it takes a long time to get confidence odds are really 50%-50%
  + But if $P(H) \gg P(T)$, we learn a lot every time the coin comes up tails. If we flip HTHHH----, the 2nd tail tells us a lot about $P(T)$
  + Information is about classical ignorance

- An alternative approach is to think about data compression
  + Think about an alphabet w/ 4 letters a, b, c, d.
    A simple binary encoding is $a=00$, $b=01$, $c=10$, $d=11$
  + But this is inefficient if some letters are uncommon, say $P(a) = 1/2$, $P(b) = 1/4$, $P(c) = P(d) = 1/8$
  + A more efficient binary representation is (actually most efficient)
    $a=0$, $b=10$, $c=110$, $d=111$
    ie, max length is 3, 0 means change letters.
  + The length of a letter $i(\text{letter}) = -\log_2(P(\text{letter}))$ is the number of Shannon bits
  + The average length of a letter in a word / conversation is the Shannon entropy

$$S = -\sum P \log_2 P$$

  + The Shannon entropy formula may be familiar from statistical mechanics, and it is in fact the same idea — we are ignorant of the microstate.

- Quantum information is also about ignorance
  + Even if we know the exact quantum state $|\psi\rangle$, there is quantum probability. This does ~~not~~ not contribute to the information content/Shannon entropy
  + If the quantum state of a system is definite (even a Schrödinger cat state), it has zero entropy and is called a pure state.
  + But suppose someone prepares a quantum state, and you don't know what it is. (This is what happens in statistical mechanics.) This is called a mixed state, and our ignorance means it contains information
  + We can think of a mixed state as choosing a (possibly special) basis of states and a probability for the actual state to be each one:

    P_1 for $|e_1\rangle$, P_2 for $|e_2\rangle$, etc

    A pure state is a mixed state where one probability $P_i = 1$ and the rest are zero

- Density Matrix (or Density Operator):
  + A pure state is mathematically described by a normalized ket. What is a mixed state?
  + We need a way to list probabilities P_i that is not a vector (b/c those represent kets).
  + But we can use a matrix. Define the density matrix in (the $\{|e_i\rangle\}$ basis) as

$$\rho \equiv \text{diag}(P_1, P_2, \cdots P_N)$$

Can change basis by similarity transformation

  + This is Hermitian $\rho^\dagger = \rho$ and has $Tr(\rho) = \sum P_i = 1$
  + We can define this as an operator

$$\rho \equiv P_1 |e_1\rangle\langle e_1| + P_2 |e_2\rangle\langle e_2| + \cdots$$

+ Each term is called a dyad and acts on a ket $|\psi\rangle$ by $\rho|\psi\rangle = \sum_i P_i (\langle e_i|\psi\rangle) |e_i\rangle$

This is equivalent to matrix multiplication of ρ on the column describing $|\psi\rangle$ in the $|e_i\rangle$ basis

+ Each dyad is a column × row = square matrix
+ A pure state $|e_1\rangle \equiv |\psi\rangle$ has density matrix

$$\rho = |\psi\rangle\langle\psi|$$

+ Suppose we want to write ρ as a matrix in another basis $\{|f_i\rangle\}$. Take the dyad form. Then the matrix elements are

$$\rho_{ij} = \langle f_i|\rho|f_j\rangle = \sum_k P_k \langle f_i|e_k\rangle\langle e_k|f_j\rangle$$

+ Examples: Suppose we know an electron has 50% chance of being spin up $|\uparrow\rangle$ or down $|\downarrow\rangle$. In the usual basis, $(|e_1\rangle = |\uparrow\rangle, |e_2\rangle = |\downarrow\rangle)$

$$\rho \triangleq \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

But if it is "spin right" $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$,

$$\rho = |\psi\rangle\langle\psi| \triangleq \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

So here we don't have $\{|\uparrow\rangle, |\downarrow\rangle\} = \{|e_1\rangle, |e_2\rangle\}$ b/c ρ is not diagonal. If we diagonalize it, the probabilities are $P_1 = 1$ and $P_2 = 0$.

- Entropy + the density matrix

+ For a diagonal matrix M with diagonal entries M_1, M_2, etc, a function $f(M) = \text{diag}(f(M_1), f(M_2) \dots)$
+ So

$$S = -\text{Tr}\, \rho \log_2(\rho) = -\sum_i P_i \log_2 P_i$$

generalizes Shannon entropy to quantum form. So we see a pure state has zero entropy. The entropy of a mixed state is our ignorance.

+ But the universe is in a pure state, presumably. How do we get a mixed state? It's b/c we only know about part of the full system

+ Simplest example: suppose we have an electron + positron in spin state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$.
But if we only have the electron, it doesn't have a well-defined pure state by itself. We can make a mixed state for the electron by "tracing" over the positron states

$$\rho_1 \equiv {}_2\langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle_2 + {}_2\langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle_2$$
$$= \tfrac{1}{2}|\downarrow\rangle_1\,{}_1\langle\downarrow| + \tfrac{1}{2}|\uparrow\rangle_1\,{}_1\langle\uparrow|$$

This is a mixed state and has entropy!

+ This is a key property of entanglement: if you start with multiple particles in a pure state and "trace out" some of them, the remaining particles are in a mixed state iff the multiparticle pure state is entangled. The entropy of the remaining particles is the entanglement entropy and measures the entanglement

- There are many other aspects of quantum information theory and it is rapidly becoming a critical subject to learn for many fields of physics