

Advanced Quantum Mechanics PHYS-4602

Final Exam

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15-18 April 2020

Instructions:

- This exam begins at 5PM Central Daylight Time Wednesday 15 April 2020.
- You may hand write and scan (or photograph if necessary) your solutions or type them as a \LaTeX document.
 - If you hand write, start each new question on a fresh page.
 - If you scan as a single file, ensure that pages are in order of the problems.
 - If you must scan or photograph your solutions as individual pages, include the page number in the file names, for example `exam-pg1.jpg`, `exam-pg2.jpg`, etc.
- This exam is **due 5PM Central Daylight Time Saturday 18 April 2020**. Submit it by **email** to `a.frey@uwinnipeg.ca`. I will reply with a confirmation email when I have saved and viewed each file.
- You may use **only** the following resources:
 - Either 2nd or 3rd edition of the Griffiths *Introduction to Quantum Mechanics* textbook
 - Your own personal notes from class lectures
 - Any item linked on the course webpage including the linked lecture notes, assignment solutions, and the Wilde article in the reading assignments.
 - Wolfram Alpha for calculations if you cite where you use it in your solutions. (To use it, enter what you want to do in the text box.)

No other resources are allowed including any other type of calculator or mathematical software.

- You may use results from the allowed resources without re-deriving them.
- You may email me at `a.frey@uwinnipeg.ca` with any questions. I will answer as soon as possible.
- You may **not** collaborate.
- This test has 2 pages of questions (4 total pages including cover sheets).
- **Answer all questions briefly and completely.**

See next page for formulae

Useful Mathematics

- Exponential Integral for $\text{Re } a > 0$

$$\int_0^{\infty} dx x^p e^{-ax} = p! a^{-p-1}$$

- Gaussian Integrals

- For $\text{Re } a > 0$

$$I_0(a) \equiv \int_0^{\infty} dx e^{-ax^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad I_1(a) \equiv \int_0^{\infty} dx x e^{-ax^2} = \frac{1}{2} \int_0^{\infty} d(x^2) e^{-a(x^2)} = \frac{1}{2a}$$

- With additional powers

$$\int_0^{\infty} dx x^{2n} e^{-ax^2} = \frac{d^n I_0}{da^n}, \quad \int_0^{\infty} dx x^{2n+1} e^{-ax^2} = \frac{d^n I_1}{da^n}$$

- For exponent $-(ax^2 + bx)$, complete the squares

$$-(ax^2 + bx) = -a \left(x + \frac{b}{2a} \right)^2 + \frac{b^2}{4a}$$

and shift integration variables to $x' = x + b/2a$

- Trig and hyperbolic trig functions

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

- Derivatives and integrals follow from these
- $1 = \cos^2 \theta + \sin^2 \theta = \cosh^2 \theta - \sinh^2 \theta$
- Angle addition

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta, & \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sinh(\alpha + \beta) &= \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta, & \cosh(\alpha + \beta) &= \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta \end{aligned}$$

- Often useful: $\sin^2 \theta = (1 - \cos(2\theta)) / 2$, $\cos^2 \theta = (1 + \cos(2\theta)) / 2$

Answer all questions briefly but completely. You may re-use results in multiple problems if helpful, but please reference the first problem where you use them.

Recent Topics

1. The Yukawa potential (in 3D) is a central potential given by

$$V(r) = -\beta \frac{e^{-\mu r}}{r} , \quad (1)$$

where β, μ are constants. This is the potential for electromagnetism with massive photons and a (very oversimplified) model for the strong force between neutrons and protons.

- (a) [5 points] Find the total cross section for scattering of a particle of mass m from the Yukawa potential in the Born approximation at low energies.

In the following parts, use a (normalized) Gaussian trial wavefunction $\psi(\vec{r}) = (2b/\pi)^{3/4} \exp(-br^2)$ to estimate the ground state energy of the Yukawa potential.

- (b) [5 points] Show that the expectation value of the kinetic energy is $\langle \vec{p}^2/2m \rangle = 3\hbar^2 b/2m$.
Hint: think about the wavefunction in Cartesian coordinates.
- (c) [10 points] Show that the potential energy satisfies

$$\langle V \rangle \leq -\beta \sqrt{\frac{8b}{\pi}} + \beta \mu e^{\mu^2/8b} . \quad (2)$$

Hints: use the Gaussian integral relations on the cover sheet. Think also about how the value of a Gaussian integral changes as the limits of integration change.

- (d) [5 points] Using your results from the previous two parts, find an upper limit for the ground state energy given a value of b . Then improve your bound for small μ by finding the optimal value of b in the case that $\mu = 0$ and substituting that in the $\mu > 0$ limit.
2. Consider a 1-dimensional harmonic oscillator with potential $V = m\omega^2 x^2/2$ in the WKB approximation.

- (a) [5 points] Show that the energy eigenvalues in the WKB approximation are equal to the exact energy eigenvalues of the harmonic oscillator.
- (b) [10 points] Consider the ground state WKB wavefunction in the classically allowed region where $E > V(x)$. Starting from the connection formula, show that the WKB wavefunction can be written as

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \cos \left[\int_0^x dx' p(x')/\hbar \right] \quad (3)$$

for constant A .

- (c) [10 points] Now write the Taylor expansion of the WKB ground state wavefunction $\psi(x)$ to second order in x . Does it match the Taylor expansion of the exact ground state wavefunction?

Continue to next page

Cumulative

3. Consider a system of 3 qubits. For notation, a subscript i on a 1-qubit operator means it acts on qubit i , so \mathbb{H}_i is the Hadamard operator acting on qubit i . We also define $\Gamma_{i,j}$ as the CNOT operator acting on qubit j with qubit i as control. For example, $\Gamma_{1,2}$ is the usual CNOT operator that reverses qubit 2 if qubit 1 is $|1\rangle$, while $\Gamma_{3,1}$ reverses qubit 1 if qubit 3 is $|1\rangle$, etc. Finally, we denote $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ for any qubit.

- (a) [5 points] Show that

$$\frac{1}{\sqrt{2}} (|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2) = \frac{1}{\sqrt{2}} (|+\rangle_1|+\rangle_2 + |-\rangle_1|-\rangle_2) . \quad (4)$$

- (b) [10 points] We wish to define a new operator $\Delta_{i,j}$ that takes $\Delta_{i,j}|+\rangle_i|0\rangle_j = |+\rangle_i|+\rangle_j$ and $\Delta_{i,j}|-\rangle_i|0\rangle_j = |-\rangle_i|-\rangle_j$. Show that $\Delta_{i,j} \equiv \mathbb{H}_i\mathbb{H}_j\Gamma_{i,j}\mathbb{H}_i$ obeys these equations. Then find $\Delta_{i,j}|+\rangle_i|1\rangle_j$ and $\Delta_{i,j}|-\rangle_i|1\rangle_j$.

- (c) [10 points] Let $|\psi\rangle = a|0\rangle + b|1\rangle$ be a general qubit. Show that

$$\Gamma_{3,2}\Delta_{1,2}\Gamma_{1,3}|\psi\rangle_1|0\rangle_2|0\rangle_3 = \frac{1}{\sqrt{2}} \left(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2 \right) |\psi\rangle_3 . \quad (5)$$

4. A particle moves in an isotropic 2D harmonic oscillator with Hamiltonian

$$H_0 = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{1}{2}m\omega^2 (x^2 + y^2) . \quad (6)$$

The eigenstates can be described by excitation numbers n_x, n_y in the two Cartesian directions in the form $|n_x, n_y\rangle$, and the energy eigenvalues are $\hbar\omega(n+1)$, where $n = n_x + n_y$. The particle is subject to a perturbing Hamiltonian

$$H_1 = \alpha \left[(a_y^\dagger)^2 a_x^2 + (a_x^\dagger)^2 a_y^2 \right] \quad (7)$$

in terms of the harmonic oscillator ladder operators in each Cartesian direction.

- (a) [10 points] Write H_1 as a sum of dyad operators of the form $|n_x, n_y\rangle\langle n'_x, n'_y|$. You can do this by comparing the operations of H_1 and the individual dyads on a state $|n''_x, n''_y\rangle$.
- (b) [15 points] Find the energy eigenvalues for the total excitation number $n = n_x + n_y = 2$ states to first order in the small parameter α .