Advanced Quantum Mechanics PHYS-4602
In-Class Test

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Instructions:

- Do not turn over until instructed.
- You will have 75 minutes to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae:

- Schrödinger Equation
  - time-dependent and position-basis time-independent

$$i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle \ , \quad \langle\vec{x}|H|\psi\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}) + V(\vec{x})\psi(\vec{x}) = E\psi(\vec{x})$$

  - Ehrenfest's Theorem and time-evolution operator

$$\frac{d\langle\mathcal{O}\rangle}{dt} = \frac{i}{\hbar}\langle[H,\mathcal{O}]\rangle + \left\langle\frac{\partial\mathcal{O}}{\partial t}\right\rangle \ , \quad U(t) = e^{-iHt/\hbar}$$

- Dirac Notation
  - Matrix element in $|e_i\rangle$ basis: $A_{ij} = \langle e_i|A|e_j\rangle$
  - Dyad representation in any basis and eigenbasis

$$A = \sum_{i,j} A_{ij}|e_i\rangle\langle e_j| = \sum_n a_n|a_n\rangle\langle a_n|$$

- Quantum Information
  - Shannon entropy $S = -\sum P\log_2 P$
  - Density operator $\rho = P_1|e_1\rangle\langle e_1| + P_2|e_2\rangle\langle e_2| + \cdots$ (its eigenbasis), $\rho = |\psi\rangle\langle\psi|$ (pure state)
  - Von Neumann entropy $S = -Tr(\rho\log_2\rho)$

- Quantum Computing
  - 1-bit gates $I : (|0\rangle \to |0\rangle, |1\rangle \to |1\rangle); NOT : (|0\rangle \to |1\rangle, |1\rangle \to |0\rangle);$
    $R(\phi) : (|0\rangle \to |0\rangle, |1\rangle \to e^{i\phi}|1\rangle); \mathbb{H} : (|0\rangle \to (|0\rangle + |1\rangle)/\sqrt{2}, |1\rangle \to (|0\rangle - |1\rangle)/\sqrt{2}$
  - 2-bit controlled NOT gate $CNOT(|x\rangle|y\rangle) = |x\rangle|x \oplus y\rangle$, $\oplus =$ addition mod 2

- Momentum and Position
  - Inner product of eigenstates (change of basis) in 1D

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \quad \text{or} \quad |p\rangle = \int dx \, \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}|x\rangle$$

    Normalization is given for infinite volume
  - Operator commutator $[x, p] = i\hbar$
  - Operator representation in position basis

$$\langle x|p|\psi\rangle = -i\hbar \frac{d\psi}{dx} \ \Leftrightarrow p \simeq -i\hbar \frac{d}{dx}$$

- Uncertainty
  - The uncertainty $\sigma_{\mathcal{O}}$ obeys $\sigma_{\mathcal{O}}^2 = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$
  - Heisenberg Uncertainty Prinicple $\sigma_A \sigma_B \geq (1/2)|\langle [A, B] \rangle|$

- Infinite Square Well $V(x) = 0$ for $-a < x < a$, $V(x) = \infty$ otherwise
  - Eigenfunctions

$$\psi_n(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right) \ (n \text{ odd}) \text{ or } \psi_n(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) \ (n \text{ even})$$

  - Energy Eigenvalue $E_n = (\hbar^2/2m)(n\pi/2a)^2$

- 1D Harmonic Oscillator $V(x) = (1/2)m\omega^2 x^2$
  - Ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}} x + i \frac{p}{\sqrt{2\hbar m\omega}} \ , \ a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \ [a, a^\dagger] = 1$$

  - Hamiltonian $H = \hbar\omega \left(a^\dagger a + 1/2\right)$

- Hydrogen
  - States are denoted $|n, \ell, m, m_s\rangle$ or $|n, j, \ell, m_j\rangle$ (recall that $s = 1/2$ always for electrons).
  - Bohr radius $a = 4\pi\epsilon_0\hbar^2/me^2$ and energy $E_n = -(\hbar^2/2ma^2)(1/n^2) = -13.6 \text{ eV}/n^2$