

## PHYS-3203 Homework 9 Due 3 April 2020

This homework is due 11PM on the due date. You may email a PDF (typed, scanned, or photographed) to Dr. Frey.

### 1. Derivatives Have Lowered Indices

As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

$$a^{\mu'} = \Lambda^{\mu'}_{\nu} a^{\nu} \quad \text{and} \quad a_{\mu'} = \Lambda^{\nu}_{\mu'} a_{\nu} , \quad (1)$$

where  $[\Lambda^{\mu'}_{\nu}]$  is the usual Lorentz transformation matrix from  $S \rightarrow S'$  and  $[\Lambda^{\nu}_{\mu'}]$  is its matrix inverse (the transformation from  $S' \rightarrow S$ ).

- Using the fact that the spacetime position  $x^{\mu}$  is a 4-vector, find the partial derivatives  $\partial x^{\mu}/\partial x^{\nu'}$  and  $\partial x^{\mu'}/\partial x^{\nu}$  in terms of  $\Lambda^{\mu'}_{\nu}$  and  $\Lambda^{\nu}_{\mu'}$ . *Hint:* For two positions as measured in the same frame,  $\partial x^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$  (think about why).
- If  $f$  is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the multivariable chain rule to show that

$$\frac{\partial f}{\partial x^{\mu'}} = \Lambda^{\nu}_{\mu'} \frac{\partial f}{\partial x^{\nu}} . \quad (2)$$

In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write  $\partial_{\mu} f \equiv \partial f / \partial x^{\mu}$ .

### 2. Some Scalar Products

In some frame, the components of two 4-vectors are

$$a^{\mu} = (2, 0, 0, 1) \quad \text{and} \quad b^{\mu} = (5, 4, 3, 0) . \quad (3)$$

*inspired by a problem in Hartle*

- Find  $a^2$ ,  $b^2$ , and  $a \cdot b$ .
- Does there exist another inertial frame in which the components of  $a^{\mu}$  are  $(1, 0, 0, 1)$ ? What about  $b^{\mu}$ ? Explain your reasoning.

Now consider lightlike 4-vectors  $f^{\mu}$  and  $g^{\mu}$ .

- If  $f^{\mu}$  and  $g^{\mu}$  are orthogonal ( $f \cdot g = 0$ ), prove that they are parallel ( $f^{\mu} \propto g^{\mu}$ ).
- Is the 4-vector  $f^{\mu} + g^{\mu}$  spacelike, timelike, or lightlike? Assume that both  $f^0 > 0$  and  $g^0 > 0$ .

### 3. SN1987A and Neutrino Masses

On 23 Feb 1987, astronomers were startled by the observation of a new supernova in the Large Magellanic Cloud, a satellite galaxy of our Milky Way. However, the first observation of this supernova was several hours earlier by the detection of neutrinos, which was confirmed by two detectors. (The neutrinos arrived before the light because light is trapped for a while by all the matter inside the exploding star.) The fact that the neutrinos all arrived within a few seconds of each other after traveling for more than 100,000 lightyears allows us to put tight constraints on the mass of the neutrino. This problem will guide you through a real calculation of this limit.

- (a) Show that a neutrino with energy  $E \gg mc^2$  has a speed approximately given by

$$\frac{|\vec{u}|}{c} \approx 1 - \frac{1}{2} \left( \frac{mc^2}{E} \right)^2 . \quad (4)$$

*Hint:* We gave formulas in class for energy both in terms of the spatial momentum and in terms of the speed. Try looking at those. Then you will need to make an expansion in powers of  $mc^2/E$ .

- (b) Light (once free of the matter in the supernova) takes a time  $t_0 = 5.3 \times 10^{12}$  s to travel from SN1987A to the earth. How long would a neutrino of energy  $E$  take to reach earth from the supernova? Work to the lowest non-trivial order in  $mc^2/E$  and give the answer in terms of  $t_0$ ,  $m$ ,  $c$ , and  $E$ . Use (4).
- (c) The Kamioka detector in Japan detected several neutrinos. The first arrived with energy 21.3 MeV, and another with energy 8.9 MeV arrived 0.303 s later. Assuming that the second neutrino left the supernova no more than 1 s before the first, what is the maximum neutrino mass  $m$ ? For simplicity, we are ignoring the possible error in the measurements. *Hint:* The observation time of each neutrino is its emission time plus its travel time; take the difference of these and be careful of signs.

For your interest, these neutrino measurements were made by a predecessor experiment to one of the experiments that led to the 2015 Nobel Prize in Physics.

#### 4. Neutrino Recoil

When a muon-neutrino and an electron collide, the neutrino and electron can transform into an electron-neutrino and a muon. For reference, the electron and muon have masses  $m_e$  and  $m_\mu$  respectively with  $m_\mu > m_e$ , and you may approximate both types of neutrino as being massless.

- (a) If the electron is initially at rest, what is the minimum initial neutrino energy required for this process to occur?
- (b) Suppose the electron is initially at rest and the final electron-neutrino moves off at an angle  $\theta$  from the initial direction of motion of the muon-neutrino. If the initial muon-neutrino energy is  $E$ , find the energy of the electron-neutrino.