PHYS-3203 Homework 8 Due 18 Mar 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Exploding Cannonball** *inspired by a problem by Barton (and other texts)*

   A cannonball is launched in an arc with velocity $\vec{u}$. At the top of its trajectory, a chemical charge in it explodes into two parts of masses m_1 and m_2 that separate in the horizontal direction only. The explosion releases energy E, which essentially all goes into the kinetic energy of the cannonball pieces. Show that they are separated by a distance $(u_y/g)\sqrt{2E(m_1+m_2)/m_1m_2}$ when they land, where u_y is the initial vertical component of the velocity.

2. **Lagrange Points** *related to KB 10.15 & 16*

   Consider the restricted 3-body problem as described in the class notes. If the radius of the primary-secondary orbit is a, the frequency of the orbit is $\omega = \sqrt{G(m_1+m_2)/a^3}$ for primary and secondary masses m_1, m_2. The distances of the primary and secondary from their center of mass are $a_1 = m_2a/(m_1+m_2)$ and $a_2 = m_1a/(m_1+m_2)$ respectively. Work in an accelerating reference frame that rotates with the primary and secondary (ie, they are at fixed positions). Assume all motion is in the xy plane.

   (a) In this frame, the force on a tertiary of mass m_3 moving in the same plane is

   $$\vec{F} = -Gm_3\left(\frac{m_1}{r_1^3}\vec{r}_1 + \frac{m_2}{r_2^3}\vec{r}_2\right) + m_3\omega^2\vec{r} - 2m_3\vec{\omega}\times\dot{\vec{r}}\ . \tag{1}$$

   where $\vec{r}$ is the position of the tertiary and $\vec{r}_1 \equiv \vec{r} - a_1\hat{x}$, $\vec{r}_2 \equiv \vec{r} + a_2\hat{x}$. Find the effective potential for a stationary tertiary (ie, corresponding to all terms in the force except for the Coriolis force). Write your answer in terms of the masses and primary-secondary orbit radius.

   (b) Sketch the effective potential for tertiary positions on the x axis for the cases $m_1 = m_2$ and $m_1 \gg m_2$. In both cases, use your sketch to argue that there are (unstable) equilibrium points between the primary and secondary as well as outside each of them.

3. **Boosts and Rotations**

   In matrix form, we can define the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

   $$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\ ,\quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}\ . \tag{2}$$

   Empty elements in the matrices above are zero.

   (a) In matrix form, the metric $\eta_{\mu\nu}$ is

   $$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\ . \tag{3}$$

Show that both rotation and boost in (2) satisfy the condition $\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta}$, which is $\eta = \Lambda^T \eta \Lambda$ in matrix notation.

(b) Consider two successive boosts along x, $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3. Using the relationship $v/c = \tanh\phi$ between velocity and rapidity ϕ, reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.

(c) First, write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction by permuting axes. Then show that you can get a boost along y by rotating axes, boosting along x, then rotating back by proving that $\Lambda_{ty}(\phi) = \Lambda_{xy}(-\pi/2)\Lambda_{tx}(\phi)\Lambda_{xy}(\pi/2)$.