# PHYS-3203 Homework 5 Due 12 Feb 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Weak Coupling as Forcing**

   Consider two objects, each of mass $m$, attached to springs of spring constants $k_1$ and $k_2$ respectively (these springs are also attached to facing walls). The objects are attached to each other by a third spring of constant $\bar{k}$, as described in the example from the lecture notes. The EOM are

$$m\ddot{x}_1 + (k_1 + \bar{k})x_1 = \bar{k}x_2 \ , \tag{1}$$
$$m\ddot{x}_2 + (k_2 + \bar{k})x_2 = \bar{k}x_1 \ . \tag{2}$$

   Consider this system with initial conditions $x_1 = a, x_2 = 0, \dot{x}_1 = \dot{x}_2 = 0$. The solution in the $\bar{k} \to 0$ limit is $x_1 = a\cos(\omega_1 t), x_2 = 0$, where $\omega_{1,2} = \sqrt{k_{1,2}/m}$.

   (a) Now allow $\bar{k}$ nonzero but $\bar{k} \ll k_1, k_2$. To a good approximation, the solution for $x_1, x_2$ must be similar to the $\bar{k} = 0$ solution. Therefore, we can assume that $x_2$ is negligible in equation (1) for $x_1$, the only change to the solution for $x_1$ is a change in the frequency. Find $\omega_1$ as an expansion in the small number $\bar{k}$ to first order.

   (b) To find the solution for $x_2$, consider $x_1$ to be a fixed function $x_1 = a\cos(\omega_1 t)$, so the right-hand side of (2) is an effective forcing term for $x_2$. Find the solution for $x_2$ with this forcing, assuming $k_2 \neq k_1$. Do not yet add a transient solution to solve the initial conditions.

   (c) The solution from part (b) does not satisfy the initial condition $x_2 = 0$. Add a transient solution (ie, a solution with no forcing), so $x_2$ satisfies the initial conditions, still assuming $k_2 \neq k_1$. Show that the solution can be written as $x_2 = A\sin[(\omega_1+\omega_2)t/2]\sin[(\omega_1-\omega_2)t/2]$ for some amplitude $A$. Since $A$ is small, the energy in the $x_2$ oscillator remains small at all times.

   (d) Repeat part (b) with $k_2 = k_1$. Show that the solution is $x_2 = (\bar{k}a/2m\omega_1)t\sin(\omega_1 t)$. How can you interpret this solution?

2. **Hanging Springs** *based on KB 11.3*

   Two identical springs of spring constant $k$ are both attached to the same object of mass $m$. The other end of the first spring is attached to a fixed support, while the other end of the second spring is attached to an object of the same mass $m$. When they are held horizontal, each spring has equilibrium length $l$. Consider instead orienting the springs so that the first hangs from the ceiling and the second hangs downward from the object in the middle. Define generalized coordinates for the masses such that $l+x_1$ is the length of the first spring and $l+x_2$ is the length of the second.

   (a) Write the potential energy of the system in terms of $x_1, x_2$. Then find the equilibrium positions $x_1^0, x_2^0$ of the masses by minimizing the potential.

   (b) Now define generalized coordinates $y_{1,2}$ that are the displacements from the vertical equilibrium, ie, $x_{1,2} = x_{1,2}^0 + y_{1,2}$. Find the Lagrangian in terms of the $y_{1,2}$ coordinates.

   (c) Find normal modes and the frequencies of oscillation. Describe the motion for each normal mode in terms of the ratio $y_2/y_1$. Use the generalized eigenvalue problem from the equations of motion.