PHYS-3203 Homework 4 Due 5 Feb 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Hamiltonian Central Force Motion** *expanded from KB12.4*

   Consider an object of mass m moving in 3D with a central conservative force of potential energy $V(r)$.

   (a) Write the Hamiltonian for this object in spherical polar coordinates.

   (b) You should see that the azimuthal angle ϕ is cyclic. Assuming motion is confined to the equatorial plane, find the effective potential for radial motion. Find the transformation of the Cartesian coordinates generated by p_ϕ. Use both these results to argue that $p_\phi = J_z$, the z component of angular momentum.

   (c) Define the square angular momentum

   $$\vec{J}^2 = m^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \ . \tag{1}$$

   Write $\vec{J}^2$ in terms of canonical momenta and show that it is conserved, even though θ is not cyclic.

2. **Liouville Theorem in Particle Accelerators** *from TM*

   Consider a linear accelerator, which accelerates bunches of electrons along the z axis. The beam initially has a circular cross section of radius R in the xy plane with uniform electron density across the circle. The transverse momenta p_x, p_y are likewise distributed uniformly over a circle of radius P centered on the origin of momentum space. As they move down the accelerator, some mechanism focuses the beam onto a transverse circle in the xy plane of half the initial radius. What is the resulting distribution of p_x, p_y, and what does this mean for the electron beam? *Hint:* The z motion is decoupled from the motion in the xy plane, so you can focus only on the distribution in xy phase space.

3. **Bead on Hoop** *from KB 11.7 & 8, also TM*

   A thin circular hoop of radius R and mass M hangs from a point on its rim (as a pendulum). Meanwhile, a bead of mass m slides frictionlessly around the hoop.

   (a) Write the Lagrangian in terms of the angle θ of the hoop's diameter from the vertical and the angular displacement ϕ of the bead around the hoop from the point on the hoop opposite the pivot (that is, the point opposite the pivot and the bead make an angle ϕ with the vertex at the center of the hoop).

   (b) Find the normal modes and frequencies for small oscillations around the $\theta = \phi = 0$ equilibrium. Describe the motion for each normal mode in terms the ratio of amplitudes ϕ/θ. *Hint:* In this case, I would suggest working at the level of the Lagrangian.