PHYS-3203 Homework 1 Due 15 Jan 2020

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Geodesic on a Cone** *KB 3.16 explained*

   A *geodesic* is the minimal length curve on a surface between two points on that surface (or possibly in a curved space). For example, we showed that a straight line segment is a geodesic on a plane, and you may know that a great circle is a geodesic on a sphere. Here we will examine geodesics on a cone with its tip at the origin and its axis of symmetry along the z axis. The surface of the cone is at a polar angle α from the z axis.

   (a) Find the relationship between the cylindrical coordinates ρ and z on the surface of the cone and show that the distance L from point (ρ_1, φ_1) to point (ρ_2, φ_2) on the cone can be written
   $$L = \int_{\varphi_1}^{\varphi_2} d\varphi \sqrt{\rho^2 + \csc^2\alpha\, \rho'^2} \tag{1}$$
   where $\rho' = d\rho/d\varphi$.

   (b) Show that a geodesic satisfies the equation
   $$\rho\rho'' - 2(\rho')^2 - \sin^2\alpha\, \rho^2 = 0\ . \tag{2}$$

   (c) Solve (2) for $\rho(\varphi)$ by changing variables to $\rho = 1/u$. Leave your solution in terms of 2 undetermined integration constants (do not find them in terms of the boundary conditions stated above). What do the integration constants describe?

   (d) As viewed from the origin, a geodesic of infinite length only spans a finite angle $\Delta\varphi$. Find $\Delta\varphi$. Explain the consistency of your answer with what we know about geodesics in the plane (ie, the limit as $\alpha \to \pi/2$).

2. **More on the Brachistochrone**

   Consider a particle moving along a brachistochrone path, written parametrically as
   $$x = a(\theta - \sin\theta)\ ,\quad y = a(1 - \cos\theta)\ , \tag{3}$$
   under the influence of gravity (without friction). Note that y increases downward.

   (a) Show that the time for the particle to move from $x = y = 0$ to the minimum of the curve at $x = \pi a, y = 2a$ can be written as
   $$T = \frac{1}{\sqrt{2g}} \int_0^\pi d\theta \sqrt{\frac{(dx/d\theta)^2 + (dy/d\theta)^2}{y}} \tag{4}$$
   and show that the time for the object to reach the bottom is $T = \pi\sqrt{a/g}$.

   (b) *from FC* Write T as an integral over x. Model the brachistochrone as the parabola $y = b_1 x + b_2 x^2$ and use Maple to find the value of b_1 that minimizes T (note that the boundary conditions on y specify b_2 in terms of b_1). You will need to carry out the T integral numerically for a range of b_1 values in units where $a = 1$. Find the optimum value of b_1 to 3 significant figures. Is T for this curve larger than the value for the brachistochrone (like it should be)? Attach a hardcopy of your Maple code. *Hint:* To find a list of integral values, you can use the following Maple command

```
intlist := [seq([b, int(f(b,x), x = 0 .. Pi, numeric)], b = .1 .. 10, 0.01)]:
```

where f is the integrand you want (we can leave off the factor of $\sqrt{g}$ out front). Then you can use the `listplot` command to get an idea of where the optimum value is before examining the list to find it specifically.

3. **Minimizing Surface Area** *adapted from a problem by TM*

   Consider a right-circular cylinder of radius r and height h.

   (a) If the volume of the cylinder is fixed to V, what is the ratio r/h that minimizes the surface area?

   (b) What happens if you try to minimize the surface area of only the cylindrical wall (ie, the surface area minus the area of the circular ends) or only the circular ends?

   (c) Finally, find the ratio if the circular ends of the cylinder are replaced by hemispheres extending inward from the top and bottom of the cylinder (note that the volume also changes). What happens if the hemispherical ends point out?