

# Advanced Mechanics PHYS-3203

## Final Exam

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20-23 April 2020

### Instructions:

- This exam begins at 5PM Central Daylight Time Monday 20 April 2020.
- You may hand write and scan (or photograph if necessary) your solutions or type them as a  $\text{\LaTeX}$  document.
  - If you hand write, start each new question on a fresh page.
  - If you scan as a single file, ensure that pages are in order of the problems.
  - If you must scan or photograph your solutions as individual pages, include the page number in the file names, for example `exam-pg1.jpg`, `exam-pg2.jpg`, etc.
- This exam is **due 5PM Central Daylight Time Thursday 23 April 2020**. Submit it by **email** to `a.frey@uwinnipeg.ca`. I will reply with a confirmation email when I have saved and viewed each file.
- You may use **only** the following resources:
  - The Kibble & Berkshire *Classical Mechanics* 5th ed textbook
  - The sections from the Hartle textbook in the library course e-reserves
  - Your own personal notes from class lectures
  - Any item linked on the course webpage including the linked lecture notes and assignment solutions
  - Wolfram Alpha for calculations if you cite where you use it in your solutions. (To use it, enter what you want to do in the text box.)

**No other resources are allowed including any other type of calculator or mathematical software.**
- You may use results from the allowed resources without re-deriving them.
- You may email me at `a.frey@uwinnipeg.ca` with any questions. I will answer as soon as possible.
- You may **not** collaborate.
- This test has 2 pages of questions (4 total pages including cover sheets).
- **Answer all questions briefly and completely.**

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**Answer all questions briefly but completely.** You may re-use results in multiple problems if helpful, but please reference the first problem where you use them.

### Recent Topics

1. *Pair annihilation* is the process in which a particle and its antiparticle annihilate to create photons. *Pair creation* is the reverse process where two photons create a particle and antiparticle. In this problem, consider pair annihilation and creation of electrons and positrons (the antiparticle of the electron), which both have mass  $m$ . Recall that photons are massless. (Use natural units.)
  - (a) [5 points] Suppose an electron and positron are both at rest when they annihilate into two photons. What are the allowed photon energies?
  - (b) [10 points] Now consider pair annihilation with the positron initially at rest and the electron moving at speed  $u = 3/5$ . What are the allowed photon energies?
  - (c) [10 points] Finally, consider pair creation of an electron-positron pair where one photon has initial energy  $m/2$ . What is the minimum energy that the other photon must have in order for pair creation to occur? The two photons hit head-on (ie, they are moving in opposite directions).
2. Imagine an introductory lab experiment in which students collide identical disks moving frictionlessly on a horizontal table. One disk is initially at rest while the other moves toward it at speed  $v$ . The disks have mass  $m$  and radius  $a$ ; they each therefore have moment of inertia  $I = ma^2/2$  around their own center. All motion is in the plane of the table.
  - (a) [10 points] Argue that the total angular momentum and kinetic energy are given by

$$\vec{J} = M\vec{R} \times \dot{\vec{R}} + \mu\vec{r} \times \dot{\vec{r}} + I\vec{\omega}_1 + I\vec{\omega}_2, \quad T = \frac{M}{2}\dot{\vec{R}}^2 + \frac{\mu}{2}\dot{\vec{r}}^2 + \frac{1}{2}I\vec{\omega}_1^2 + \frac{1}{2}I\vec{\omega}_2^2, \quad (1)$$

where  $M$  is the total mass,  $\mu$  is the reduced mass,  $\vec{R}$  is the overall CM position,  $\vec{r}$  is the relative separation, and  $\vec{\omega}_{1,2}$  are the angular velocities of the two disks. *Hint:* First change to the overall CM frame, then break the motion of each disk into its own CM motion and motion around its own CM. Note that the formula for angular momentum is similar to one we used but did not derive in our discussion of tidal friction.

- (b) [10 points] Suppose the disks collide elastically. In the laboratory frame, the initially moving disk scatters by an angle  $\pi/4$ . By converting to the CM frame, find the final velocities of both disks in the lab frame. Note that the disks cannot start rotating because there is no friction between them in an elastic collision.
- (c) [5 points] Consider an inelastic collision with friction that causes the two disks to spin with opposite angular velocities of magnitude  $\omega$  after the collision. Accounting for angular momentum conservation, what is the maximum allowed  $\omega$  assuming initial impact parameter  $b$ ? *Hint:* Due to friction, the final kinetic energy is less than the initial kinetic energy. Also, the final impact parameter may differ from the initial one but is always less than  $2a$  for a collision to occur. Use equation (1).

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## Cumulative

3. The action for a free relativistic particle can be written in a particular inertial frame as

$$S = -m \int_{t_1}^{t_2} dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} \quad (2)$$

(in natural units). Here  $\vec{x}$  is the spatial position and  $t$  is the time as measured in that inertial frame. We evaluate the action between times  $t_1$  and  $t_2$ .

- (a) [10 points] Find the canonical momentum and equation of motion. Show that the canonical momentum is the relativistic momentum as defined in class.
- (b) [10 points] Find the Hamiltonian and show that it is the relativistic energy (rest mass plus kinetic energy). *Hint:* you will need to write the relativistic gamma factor as a function of the momenta.
- (c) [5 points] Change integration variables to  $\tau$ , the proper time along the particle's path. With this integration variable, show that the integrand is 1, so  $S = -m\Delta\tau$  with  $\Delta\tau$  the total proper time between  $t_1$  and  $t_2$ . Assume that  $t$  is a monotonically increasing function of proper time. This shows that the action is independent of reference frame.
4. A small object of mass  $m$  slides (frictionlessly) on the surface

$$z = \left(\frac{3}{l_1} + \frac{1}{l_2}\right)x^2 + \left(\frac{1}{l_1} + \frac{3}{l_2}\right)y^2 + 2\sqrt{3}\left(\frac{1}{l_1} - \frac{1}{l_2}\right)xy \quad (3)$$

under the influence of gravity.

- (a) [5 points] Write the Lagrangian for motion of this object in terms of the generalized coordinates  $x$  and  $y$ , assuming the object is pointlike.
- (b) [15 points] In the case of small oscillations around the minimum of the potential (at  $x = y = 0$ ), find the normal coordinates and normal mode frequencies.
- (c) [5 points] Suppose the object is a square of side  $a$  that is still small enough to fit snugly to the surface but large enough that we should treat it as a composite object. The square has uniform density. Can the potential energy be written in terms of its center of mass position in the  $x, y$  plane? Give a brief calculation explaining your answer.