Intermediate Mechanics PHYS-3202
In-Class Test

Dr. Andrew Frey

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Instructions:

- Do not turn over until instructed.
- You will have 50 minutes to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae:

- Lagrangian Mechanics
  - Action and Lagrangian
  $$S = \int_{t_1}^{t_f} dt\, L(q, \dot{q}, t)\ , \quad L = T - V$$
  - Euler-Lagrange Equations
  $$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$
  - For constraint $f(q, \dot{q}, t) = 0$, add Lagrange multiplier term $\Delta L = \lambda f(q, \dot{q}, t)$
  - Framework applies to other optimization problems
- Hamiltonian Mechanics
  - Canonical Momentum $p_i = \partial L/\partial \dot{q}_i$
  - Hamiltonian as Legendre transform
  $$H(q, p) = \sum_i p_i \dot{q}_i - L(q, \dot{q}) \text{ with } \dot{q}_i \equiv \dot{q}_i(q, p)$$
  - Hamilton's equations $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$
  - Transformation of F generated by G is $\delta F = \{F, G\}\delta\lambda$ with Poisson bracket
  $$\{F, G\} \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$
  - Time dependence $dF/dt = \partial F/\partial t + \{F, H\}$

  - Generator from Noether's theorem $G = \sum_i (\partial L/\partial \dot{q}_i)\delta q_i$
  - Liouville theorem $d\rho/dt = 0$ along phase space trajectory for phase space density ρ
  - Virial theorem $\langle T \rangle = -(1/2)\langle \sum \vec{F} \cdot \vec{x} \rangle = (n+1)\langle V \rangle/2$ for $V \propto r^{n+1}$

- Coupled Harmonic Oscillators
  - Lagrangian
$$L = \sum_{i,j} \left(\frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} V_{ij} q_i q_j \right)$$
  - Generalized eigenvector problem for normal modes $(V - m\omega^2)B = 0$
    B =normal mode vector, V, m =matrices from Lagrangian
  - Normal coordinates $\ddot{\eta}_n + \omega_n^2 \eta_n = 0$, $L = (1/2)\sum_n (\dot{\eta}_n^2 - \omega_n^2 \eta_n^2)$
  - In forced case, determine forcing on each normal coordinate

- Waves on a String
  - Normal modes of light string with uniformly spaced identical beads (ends fixed)
$$y_{j,n} = a \sin\left(\frac{jn\pi}{N+1}\right) \ , \quad \omega_n^2 = \frac{4F}{m\ell} \sin^2\left(\frac{n\pi}{2(N+1)}\right) \ , \quad n = 1, 2, \cdots N$$
  - Normal modes of massive string F =tension, μ =mass density, $y(0,t) = y(L,t) = 0$
$$y_n(x) = \sin\left(\frac{n\pi}{L}x\right) \ , \quad \omega_n = \frac{n\pi v}{L} \ , \quad v = \sqrt{F/\mu}$$
  - Wave equation $\ddot{y} - v^2 y''$
    * Solution $y = f(x + vt) + g(x - vt)$
    * With Dirichlet b.c. at $x = 0, L$, $g(u) = -f(-u)$, $f(u + 2L) = f(u)$
    * $y(x, 0) = f_-(x)$, $\dot{y}(x, 0) = v f'_+(x)$, $f_\pm(u) = f(u) \pm f(-u)$
  - Phase velocity ω/k, group velocity $d\omega/dk$

- Velocities in Possible General Coordinates
  - Cylindrical coordinates $x = \rho\cos\varphi, y = \rho\sin\varphi, z = z$, $\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z}$
  - Spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$
    $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$