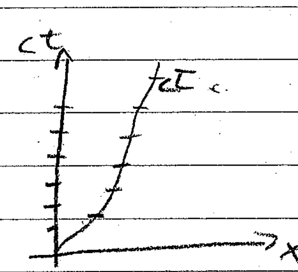
# Momentum + Applications

## — 4-Velocity

- Path of a particle in spacetime




  + One way to describe the trajectory of an object is spatial position as a function of time in a specific inertial frame $x^i(t)$ or $\vec{x}(t)$.
  + This leads to a velocity $\vec{u} = d\vec{x}/dt$. The trouble is that this velocity is not simply related to $\vec{u}' = d\vec{x}'/dt'$, the velocity in another frame. If $S'$ is moving at $v$ along $\hat{x}$ w.r.t. $S$,

$$u_x' = \frac{u_x - v}{1 - v u_x/c^2}, \quad u_y' = \frac{1}{\gamma(v)}\frac{u_y}{1 - v u_x/c^2}, \quad u_z' = \frac{1}{\gamma(v)}\frac{u_z}{1 - v u_x/c^2}$$

  + The problem is $\vec{u}$ is made in a way that isn't obviously relativistic.
  + Describe the particle's path instead as the spacetime 4-vector position $x^\mu$ as a function of proper time $\tau$ measured along the worldline or path of the particle.
  + Because $\tau$ is Lorentz invariant, we define a 4-vector velocity (4-velocity)

$$U^\mu \equiv dx^\mu/d\tau$$

- Properties of 4-velocity
  + Lorentz transformation is

$$U^{\mu'} = \frac{d}{d\tau}(x^{\mu'}) = \frac{d}{d\tau}\left(\Lambda^{\mu'}{}_{\nu} x^\nu\right) = \Lambda^{\mu'}{}_{\nu}\frac{dx^\nu}{d\tau} = \Lambda^{\mu'}{}_{\nu} U^\nu$$

  ie, 4-vector transformation

  + The proper time is

$$d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu/c^2$$

or

$$d\tau = dt\sqrt{1 - \vec{u}^2/c^2}$$

+ The components are

$$U^0 = \frac{c\,dt}{d\tau} = \frac{c}{\sqrt{1-\vec{u}^2/c^2}} = c\gamma(\vec{u}), \quad U^i = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma(u)\,u^i$$

Note: $U^0, |\vec{U}| \to \infty$ as $|\vec{u}| \to c$

This is a sign of the "universal speed limit"

+ The coordinate velocity is $u^i/c = U^i/U^0$

+ Any (massive) object has

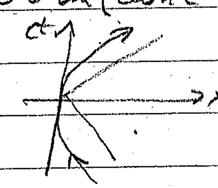
$$U^2 = \eta_{\mu\nu} U^\mu U^\nu = (\eta_{\mu\nu}\,dx^\mu dx^\nu)/d\tau^2 = -c^2$$

(write in terms of components also) even if very fast

So $U^\mu$ is always timelike with fixed square

+ An object traveling into the future has $U^0 > 0$.

• Example: Consider a particle on path $x(t) = ct_0 \ln(\cosh(t/t_0))$




+ Velocity $u(t) = dx/dt = c\tanh(t/t_0)$
by chain rule, $u(0) = 0$, $u(t \to \pm\infty) \to \pm c$

+ The proper time is

$$\tau = \int dt\,\sqrt{1-u^2/c^2} = \int dt/\cosh(t/t_0)$$

The solution is given by

$$\tan(\tau/2t_0) = \tanh(t/2t_0)$$

+ At $t \approx 0$, where $u \approx 0$, $\tau \approx t$.
At $t \to \pm\infty$ where $|u| \to c$, $\tau \to \pm\pi t_0/2$ (frozen)

+ You can then get $t(\tau)$ and $x(\tau)$ by double-angle formulas:
($t(\tau)$ from above, $x(\tau) = \ldots = -ct_0 \ln[\cos(\tau/t_0)]$)

$$U^0 = c\,dt/d\tau = c/\cos(\tau/t_0), \quad U^1 = dx/d\tau = c\tan(\tau/t_0)$$

— Acceleration + Force

• Newton's 2nd law in relativity should
  + Be covariant, so it holds in all inertial frames.
  + reduce to the usual Newton's law when all speeds $\ll c$
  + Give constant velocity $dU^\mu/d\tau = 0$ absent force

- Start defining the 4-acceleration $A^\mu \equiv dU^\mu/d\tau$
  + Since $U^2 = -c^2$ always, $d(U^2)/d\tau = 0$

    $\Rightarrow A_\mu U^\mu = 0$ ← velocity + acceleration of a particle are always orthogonal
  + From our example,

    $$A^0 = (c/t_0)\sec(\tau/t_0)\tan(\tau/t_0), \quad A^1 = (c/t_0)\sec^2(\tau/t_0)$$

    then

    $$A_\mu U^\mu = (c^2/t_0)\sec(\tau/t_0)\left[-\tan(\tau/t_0)\sec(\tau/t_0) + \sec(\tau/t_0)\tan(\tau/t_0)\right] = 0.$$

    You can also work through the examples in Hartle.

- Set the 2nd law to be $mA^\mu \equiv F^\mu$
  + We will see what this means next
  + The orthogonality property means that

    $$U^0 F^0 = U^i F^i \Rightarrow F^0 = F^i u^i/c$$
  + To be precise, $m$ is the rest mass, ie, the mass measured in the object's rest frame. This is a scalar (invariant) quantity, so the 2nd law is covariant

— 4. Momentum

- Definitions for a 4-vector momentum
  + As a start, take $p^\mu \equiv mU^\mu$ for $m$ = rest mass
  + In components

    $$p^0 = \gamma mc, \quad p^i = \gamma m u^i$$
  + In the nonrelativistic limit,

    $$p^i = mu^i(1 + \tfrac{1}{2}u^2/c^2 + \cdots), \quad p^0 = mc + \tfrac{1}{2}mu^2/c + \cdots = \tfrac{1}{c}\left[mc^2 + \tfrac{1}{2}mu^2 + \cdots\right]$$

    $p^0$ is apparently $E/c$, where $E$ = rest energy $mc^2$ plus KE plus corrections

- Energy and momentum
  + For a single particle

    $$p^2 = p_\mu p^\mu = m^2 U^2 = -m^2c^2$$ ← even when moving close to $c$, $E \gg mc^2$

    so the square gives the rest mass. Very important
  + We already have $E = \gamma mc^2$ from above

Also

$$p_\mu p^\mu = (-E/c)(E/c) + p^i p^i = \frac{1}{c^2}(-E^2 + \vec{p}^2 c^2) = -m^2c^2$$

This is

$$E^2 = m^2c^4 + \vec{p}^2 c^2 \quad \leftarrow \text{Also very important!}$$

+ For multiple particles, $p^i = p_1^i + p_2^i + \dots + p_n^i$ and
$p^0 = E_{tot}/c = p_1^0 + \dots + p_n^0 + V/c$ with potential
We can define the invariant mass of the total $p^\mu$ by
$$p^2 = -M^2c^2$$
This would give the rest mass of a bound particle for ex.
+ Since chemical energies $V \ll mc^2$ for electrons mass is $\approx$ conserved in chemistry. But ~~it~~ it is not completely negligible in nuclear physics

• Relation to Newton's Law
+ We can write this as $dp^\mu/d\tau = F^\mu$
+ The space components are
$$(dt/d\tau)(dp^i/dt) = F^i \Rightarrow F^i = \gamma f^i$$
where $\vec{f}$ is the nonrelativistic force and $\gamma$ is the gamma factor of the particle
+ Then
$$F^0 = \gamma \vec{f}\cdot\vec{u}/c = \gamma\left(\frac{dE}{dt}\right)/c = \frac{1}{c}\frac{dE}{d\tau} = \frac{dp^0}{d\tau}$$
which checks out

• Massless Particles
+ So far everything is about normal objects. But we can take the limit $p^2 = p_\mu p^\mu = -m^2c^2 \to 0$

+ Note this does not mean $p^\mu = 0$ but $E = |\vec{p}|c$

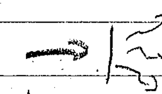
+ We also get the limit $|\vec{u}/c| = |\vec{p}|/p^0 \to 1$
so massless particles always move at light speed
These are photons, gravitons, etc

+ Note that $p^\mu$ is light like for a massless particle

- Natural Units
  - Many factors of $c$ have appeared in our notes
    + Examples $p^0 = E/c$, $p^2 = -m^2c^2$,
      $E^2 = (mc^2)^2 + (\vec{p}c)^2$, ...
    + This is because length and time have different units in SI
    + But we can think of $c_{SI} \approx 2.998 \times 10^8 m/s$ as a conversion factor rather than fundamental quantity
  - In the examples below, we will work in natural units with $c \equiv 1$
    + Length & time have same dimensions $[L] = [T]$
      This is like measuring time in years and length in light years
    + This also means $[E] = [p] = [m]$
      so
      $$p^2 = -m^2, \quad p^0 = E, \quad E^2 = \vec{p}^2 + m^2, \ldots$$
    + This is common in particle physics with energies & masses measured in eV (or MeV, GeV, etc)
    + If you need SI or similar units, you can work in natural units and insert factors of $c$ as needed by dimensional analysis
  - There are other physical constants that we can set to 1 as conversion factors. What are some?

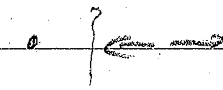
- Collisions
  - Note on reference frames
    + It is often convenient to work in the center of momentum (CM) frame, which has total $p^i = 0$.
      Becomes center of mass frame when nonrelativistic
    + Variables in CM frame are often labeled with $*$
    + In CM frame, the total 4-momentum satisfies
      $$P_{tot}^2 = -(E^*)^2 + \vec{p}^2 = -(E^*)^2$$

Since $P^2$ is a relativistic invariant, we can find the CM frame energy w/ information from any frame!

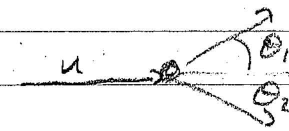
+ CM frame is the initial rest frame of a 1-particle decay or the frame of a 2-particle collision w/ equal but opposite momenta
+ Other interesting frames are a fixed-target lab frame  or the rest frame of the cosmic fluid in the early universe

• In 4-momentum (ie energy + momentum) conservation, we want to use invariants/scalar products as much as possible. Can sometimes also help to switch frames

• Example: Decay at rest to 2 particles

+ Particle of mass $M$ decays to particles of mass $m_1 + m_2$. What is energy $E_1^*$ of mass $m_1$ particle in the initial rest frame? 
+ You could try separate $E + p$ conservation.
  Start noting $\vec{p}_1^* = -\vec{p}_2^*$ with magnitude $p^*$.
  Then
  $$M = \sqrt{(p^*)^2 + m_1^2} + \sqrt{(p^*)^2 + m_2^2}.$$ Lots of algebra.
  Or note
  $$E_2^* = \sqrt{(p^*)^2 + m_2^2} = \sqrt{(E_1^*)^2 - m_1^2 + m_2^2}$$
  So
  $$M = E_1^* + E_2^* \Rightarrow (M - E_1^*)^2 = (E_1^*)^2 - m_1^2 + m_2^2$$
  $$\Rightarrow E_1^* = (M^2 + m_1^2 - m_2^2)/2M$$
  There is still some cancellation. What's an easier way?
+ Instead, try 4-vector scalar products
  $$P^\mu = p_1^\mu + p_2^\mu \Rightarrow p_2^2 = (P - p_1)^2$$
  This is
  $$m_2^2 = M^2 + m_1^2 + 2P\cdot p_1$$
  In CM frame, $P\cdot p_1 = -ME_1^*$ b/c $P^i = 0$, $P^0 = M$.
  This immediately gives the same energy.

- Example: Decay to 2 bodies in motion
  - + A $\pi^0$ particle (mass $m$) moves at speed $u$ (coordinate) w.r.t. the lab decays into 2 photons that move at angles $\theta_1$ + $\theta_2$ to the initial flight path. How does the energy $E_1$ of the 1st photon depend on $\theta_1$?
  - + Can try separate $E$ + $p$ conservation.




    Recall that photon momentum magnitude $= E$.
    Energy conservation is $\gamma m = E_1 + E_2$
    Along the velocity, momentum is $\gamma m u = E_1 \cos\theta_1 + E_2 \cos\theta_2$
    $\perp$ velocity, we have $E_1 \sin\theta_1 = E_2 \sin\theta_2$
    We can 1st eliminate $\theta_2$, then $E_2$, so

$$(\gamma m - E_1)^2 = (\gamma m u)^2 + E_1^2 - 2\gamma m u E_1 \cos\theta_1$$

    After simplifying,

$$E_1 = \frac{m}{2\gamma(1-u\cos\theta_1)}$$

  - + With 4-momentum conservation, use $p_2^\mu = P^\mu - p_1^\mu$
    so the square is $0 = -m^2 - 2P\cdot p_1$
    Note $P^\mu = (\gamma m, 0, 0, \gamma m u)$ and $p_1^\mu = (E_1, 0, E_1\sin\theta_1, E_1\cos\theta_1)$
    so

$$m^2 = 2\gamma m E_1(1-u\cos\theta_1) \quad \text{as above.}$$

  - + This result shows relativistic beaming in that, when $u$ gets large, energy almost all goes into forward-directed ($\theta_1 \approx 0$) photons b/c $\gamma$ makes denominator large when $\theta_1$ is not small.

- Example: Production Thresholds
  - + 2 particles collide. What's the minimum energy needed to create some other set of particles? Very important in experiment design.
  - + Start by considering the collision from CM frame with particles $A$ + $B$ incoming and particles $\{n\}$ outgoing.

$$-(p_A+p_B)^2 = (E_A^* + E_B^*)^2 = -(\Sigma p_n)^2 = (\Sigma E_n^*)^2 \geq (\Sigma m_n)^2$$

$$\text{so} \quad E_A^* + E_B^* \geq \Sigma m_n$$

  - + Fixed target lab frame with $A$ moving and $B$ stationary.
  - + Then

$$-(\Sigma p_n)^2 = -(p_A+p_B)^2 = +m_A^2 + m_B^2 - 2p_A\cdot p_B$$

But $p_B^0 = m_B$, $p_B^i = 0$, so $p_A \cdot p_B = -m_B E_A$

So

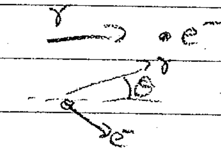
$$E_A = \frac{1}{2m_B}\left[-(\Sigma p_n)^2 - m_A^2 - m_B^2\right] \geq \frac{1}{2m_B}\left[(\Sigma m_n)^2 - m_A^2 - m_B^2\right]$$

In the inequality, we used the Lorentz invariance of $(\Sigma p_n)^2$ to an inequality derived in the CM frame.

+ This shows why colliders that want to produce heavy particles (like LHC) use CM collisions b/c $E_A^* + E_B^* \geq \Sigma m_n$, not $(\Sigma m_n)^2 \leq E_A$.

- Example: Compton Effect

+ A photon strikes a stationary electron. For low $E$, frequency is unchanged. Relativistically, the electron recoil changes photon wavelength.




+ Label initial + final electron momenta as $p^\mu$, $p'^\mu$ and initial + final photon momenta as $q^\mu$, $q'^\mu$.
+ We don't care about the final electron, so write conservation as $p'^\mu = p^\mu + q^\mu - q'^\mu$.

This squares to

$$-m^2 = -m^2 + 2p\cdot(q-q') - 2q\cdot q'$$

+ In our frame,

$$p^0 = m,\ p^i = 0;\ q^0 = E,\ q^1 = q^2 = 0,\ q^3 = E;$$
$$q'^0 = E',\ q'^1 = 0,\ q'^2 = E'\sin\theta,\ q'^3 = E'\cos\theta.$$

Then we get

$$m(E - E') = EE'(1-\cos\theta)$$

+ In quantum mechanics, photon energy is $E = hc/\lambda$ for Planck constant $h$ and wavelength $\lambda$. With simplification, the change in wavelength is

$$\lambda' - \lambda = (h/mc)(1-\cos\theta)$$

where $(h/mc)$ is the electron Compton wavelength.