Special Relativity

Galilean Relativity

- We've talked a lot about changing reference frames.
  - In particular, the laws of physics are the same in every inertial frame (Relativity Principle). In other words, there are no fictitious forces in inertial frames.
  - This means any transformation between inertial reference frames is a symmetry of the Hamiltonian or Lagrangian as in Noether's theorem, at least the generalized version.
  - We talked about generators of infinitesimal transformations previously. We'll talk about larger ones. The transformations between inertial frames in Newtonian mechanics are called Galilean transformations

- Classifying Galilean transformations
  - Translation in space $\vec{r}\,' = \vec{r} + \vec{a}$
    Shifts the origin
  - Translations in time $t' = t + b$
    Changes the origin of time
  - Rotations of axes
    + Can write in matrix form $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
    + If ~~we~~ use indices $x^1 = x$, $x^2 = y$, $x^3 = z$, this matrix multiplication is
      $$x^{i'} = R^{i'}{}_j \, x^j$$
    + Note that the repeated index on the RHS is summed over $\left(\sum_j\right)$. Einstein summation convention
  - Boosts = change in velocity
    $$\vec{r}\,' = \vec{r} + \vec{v}t$$
    + This looks like a time-dependent translation. But a general time dependence leads to acceleration
    + Really, this is more like a rotation but not quite. Why?

$$x' = \cos\theta\, x + \sin\theta\, y \qquad x' = x + vt$$
$$y' = -\sin\theta\, x + \cos\theta\, y \qquad t' = t$$

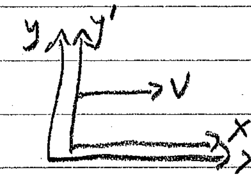
- The transformation of other quantities can be derived from these

Lorentz transformations

- Why look beyond Galilean relativity?
  - Maxwell's equations
    + In differential form, imply that there are electromagnetic waves of speed $c = 1/\sqrt{\epsilon_0 \mu_0}$ — Light
    + But they don't specify a reference frame, so what is the speed with respect to?
    + It was believed there is an <u>ether</u> permeating space providing a rest frame
    + But it's quite difficult to rewrite Maxwell's eqns in a Galilean way w/ an ether
  - Conflicting (?) experiments
    + Aberration of starlight: apparent change in position of stars due to motion of earth around the sun (like rain on a windshield)
    + Michelson-Morley experiment looking for the speed of light along different arms of a beam splitter. Found nothing
    + In ether theory, these have opposite interpretations

- Postulate:
  - Light does not need to move through a substance
  - Relativity principle applies to Maxwell's eqns $\Longrightarrow$ light always moves at $c = 1/\sqrt{\epsilon_0 \mu_0}$ (in vacuum) relative to <u>any</u> inertial frame
  - This has remarkable consequences for the structure of space + time

- Lorentz Boosts
  - Consider 2 inertial frames with S' moving at v along x w.r.t. S



- By rotational symmetry, coordinates $\perp$ to the motion must be unchanged $y' = y$, $z' = z$ (we should not be able to tell moving toward $+x$ from $-x$)

- Assume the spatial origins coincide at $t = t' = 0$.
  + The transformation of positions must be linear
  $$x' = \gamma(v)(x - vt), \quad x = \gamma(-v)(x' + vt')$$
  for motion to be inertial in both frames.
  + This ensures origin of S' is at $x = vt$ and origin of S is at $x' = -vt'$
  + $\gamma(v) = \gamma(-v)$ again by rotational symmetry

- Suppose there is a flash of light at the coinciding origin $x = 0, t = 0$ and $x' = 0, t' = 0$
  + The light ray traveling in the $+x$ direction must follow the path $x = ct$ in S and $x' = ct'$ in S'
  + Therefore
  $$ct' = \gamma(ct - vt) \quad \text{and} \quad ct = \gamma(ct' + vt')$$
  or
  $$ct' = \gamma(1 - v/c)\gamma(1 + v/c)(ct')$$
  $$\Rightarrow \gamma = 1/\sqrt{1 - v^2/c^2} \quad \text{relativistic } \gamma \text{ factor}$$
  + Also,
  $$t' = \frac{1}{v}(x/\gamma - x') = \frac{1}{v}\left[x\left(\frac{1}{\gamma} - \gamma\right) + \gamma vt\right]$$
  $$= \gamma(t - vx/c^2)$$
  and
  $$t = \gamma(t' + vx'/c^2)$$

- The Lorentz boost transformation along x is given by

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \quad \text{for } S \to S' \end{aligned} \qquad \begin{aligned} t &= \gamma(t' + vx'/c^2) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \quad \text{for } S' \to S \end{aligned}$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$

- Immediate consequences include time dilation + length contraction
  Space + time are a single concept, distinct notions of simultaneity

- Position 4-vector and Lorentz transformations
  - It would be useful to condense the Lorentz boosts into a matrix form like rotations
  - We can define 4-vectors for space-time coordinates
    - Let ct be the zeroth coordinate

$$x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

    + Label individual components with Greek superscript

$$x^\mu \Rightarrow x^0 = ct,\; x^1 = x,\; x^2 = y,\; x^3 = z$$

A Roman letter superscript is just a space coordinate

  - A Lorentz boost is then
    + $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$ (summed) in the form
      (S' frame) = (boost) · (S frame)
    + That's index notation for the matrix multiplication
      $x' = \Lambda x$ where Λ has elements $\Lambda^{\mu'}{}_\nu$.
    + The boost along x is the matrix

$$\Lambda = \begin{array}{c} \\ \mu' \downarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \begin{array}{c} \nu \rightarrow \; 0 \quad\; 1 \quad\; 2 \quad 3 \\ \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{array}$$

You can write out the summation and get the boost we wrote before.

    + This is like a rotation in the (tx) plane except for a sign. Look further:

$$\gamma^2 - \gamma^2 v^2/c^2 = (1 - v^2/c^2)/(1 - v^2/c^2) = 1$$

so we can define the rapidity ϕ such that

$$\gamma = \cosh\phi \Rightarrow \gamma v/c = \sinh\phi$$

  - Rotations also fit in the Lorentz matrices
    + Note that a 4-vector contains a regular spatial vector x^i. That is, we can think of $x^\mu = (x^0, x^i)$

+ A spatial rotation can therefore be written in 4D form as
$\Lambda^0{}_0 = 1,\ \Lambda^0{}_i = 0$
$\Lambda^i{}_0 = 0,\ \Lambda^i{}_j = R^i{}_j$ or $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$ as a block matrix.

+ To get the matrix for a boost in another direction, 1st rotate the x axis to that direction, boost along the new x, then rotate back to the original orientation

$$\Lambda_{new} = \Lambda_R^{-1} \Lambda_{old} \Lambda_R$$

- General definition of Lorentz Transformations
  + Rotations are often defined as orthogonal matrices

$$R^T R = 1$$

(This can include reflections; pure rotations also have $\det(R) = +1$)

+ What is the analog for boosts? Look at boost along x
A quick check shows

$$\Lambda^T \Lambda = \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \\ & & \ddots \end{bmatrix} \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \\ & & \ddots \end{bmatrix} \neq 1$$

Try flipping signs

$$\begin{bmatrix} -\cosh\phi & -\sinh\phi \\ \sinh\phi & \cosh\phi \\ & & \ddots \end{bmatrix} \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \\ & & \ddots \end{bmatrix} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

+ So we define a metric matrix $\eta = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$.

Then $\Lambda^T \eta \Lambda = \eta$. Since rotations are in the lower right diagonal block, this works for rotations too.

+ Lorentz transformations can be defined by matrices that satisfy $\Lambda^T \eta \Lambda = \eta$
(To exclude reflections, set $\det(\Lambda) = +1$)
These include rotations. Poincaré transformations include translations.

+ This relationship shows that $\Lambda^{-1} = \eta^{-1} \Lambda^T \eta$
For components of $\Lambda\ (S \to S') \Rightarrow \Lambda^{\mu'}{}_{\nu}$,
the inverse

$$\Lambda^{-1}(S' \to S) \text{ has } \Lambda^{\mu}{}_{\nu'} = \eta^{\mu\alpha} \Lambda_{\alpha}{}^{\beta'} \eta_{\beta'\nu'}$$

b/c transposing reverses rows + columns.

- General 4-Vectors and Invariants
  - We can define any 1-index object a^μ as a 4-vector if it has transformation $a^{\mu'} = \Lambda^{\mu'}{}_{\nu} a^\nu$

- Invariants / Scalars
  + Spatial vectors have a scalar product
    $$a^T b = a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3 \equiv a \cdot b$$
    These are invariant under rotations because
    $$a' \cdot b' = (a')^T b' = (a^T R^T)(R b) = a^T (R^T R) b = a^T b = a \cdot b$$
    since $R^T R = \mathbb{1}$.

  + What is the analogous relativistic statement? We know that $\Lambda^T \eta \Lambda = \eta$. Try $a \cdot b \equiv a^T \eta b$.
    Then
    $$a' \cdot b' = ((a')^T \eta b' = (a^T \Lambda^T) \eta (\Lambda b) = a^T (\Lambda^T \eta \Lambda) b$$
    $$= a^T \eta b = a \cdot b$$
    This checks out! This is the relativistic scalar product
  + The index notation is more common in gravitational physics
    $$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = \underset{\text{sign!}}{-}a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$
    Note that the double sum collapses to one sum b/c the metric is diagonal.
  + Use of invariants will be key in understanding physical phenomena including momentum conservation.
  + The relativistic dot product of a vector with itself is the square
    $a^2 > 0$ spacelike, $a^2 = 0$ lightlike, $a^2 < 0$ timelike
    (This is related to a sign choice)

  + The invariant interval of displacement dx^μ is the square,
    $$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^i)^2$$
    For $ds^2 > 0$, the displacement is spacelike
    and ds^2 = square of proper distance
    For $ds^2 = 0$, the spacetime points are lightlike separated ie connected by a light ray.

For $ds^2 < 0$, $ds^2 = -c^2 d\tau^2$ where $d\tau$ = proper time between the spacetime points. This is timelike separated.

° Lowered indices.

+ For the dot product to remain invariant, note that a^μ and $(\eta_{\mu\nu} b^\nu)$ must have inverse transformations
+ So define a <u>lowered index</u> 4-vector $b_\mu \equiv \eta_{\mu\nu} b^\nu$ (recall sum)
  This is $b_0 = -b^0$, $b_1 = b^1$, $b_2 = b^2$, $b_3 = b^3$.
+ What's the transformation law?

$$b_{\mu'} = \eta_{\mu'\nu'} b^{\nu'} = \eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\alpha} b^{\alpha} = (\eta_{\mu'\nu'} \Lambda^{\nu'}{}_{\alpha} \eta^{\alpha\rho}) b_\rho$$

since $\eta^2 = 1$. But we've identified the quantity in $(\cdots)$ as $\Lambda^{\mu}{}_{\nu'}$, the inverse transformation. So

$$b_{\mu'} = \Lambda^{\nu}{}_{\mu'} b_\nu$$

+ Therefore, we can always write the scalar product as

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu$$

+ We say a summed upper + lower pair is <u>contracted</u> b/c it transforms trivially. (Lorentz matrices are removed like a contraction).
+ An equation is <u>covariant</u> if the free indices are the same in every term. Then the Lorentz transformation is the same for every term, so the equation is manifestly true in every inertial frame if it is true in one.

  Examples:

$$a_\mu = \eta_{\mu\nu} b^\nu, \quad a^\mu = b^\mu$$

$$(a^\mu b_\mu) a_\nu = d_\nu, \quad \ldots$$

+ We can raise indices using the inverse metric $[\eta^{\mu\nu}] = [\eta_{\mu\nu}]^{-1}$:

$$b^\mu = \eta^{\mu\nu} b_\nu$$