Systems of Particles

General Principles

- Relativity Principle (reminder)
  - Physics is the same in all inertial reference frames, ie, forces in Newton's laws have a physical origin
  - We can choose a convenient reference frame to work in. We're now going to mostly pay attention to the relation between a general frame + the rest frame of the center of mass of an object or group of particles.

- Reminder on changing frames: Simple Procedure
  - Work out change of position coordinates
    Ex $\vec{r}' = \vec{r} + \vec{v}t$ for prime frame moving at $\vec{v}$
  - Then differentiate, etc to find transformation of other variables
  - Should be familiar from last term

The Center of Mass Frame for Many Particles

- Center of Mass (CM) and Momentum
  - Suppose we have a set of particles of masses m_i and positions $\vec{r}_i$
    + The total mass $M = \sum_i m_i$
    + The center of mass position is the mass-weighted average
      $$\vec{R} = \left(\sum_i m_i \vec{r}_i\right)/M$$
    + We can convert these to integrals for continuum materials

  - Total momentum
    + Is $\vec{P} = \sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$ = momentum of a particle mass M moving like the CM
    + Also,
      $$\dot{\vec{P}} = \sum_i \left(\vec{F}_{i,ext} + \sum_j \vec{F}_{ij}\right) = \vec{F}_{ext} + \sum_{ij} \vec{F}_{ij} = \vec{F}_{ext}$$
      total external force

      b/c $\vec{F}_{ij} = -\vec{F}_{ji}$ by 3^{rd} law.

+ In the absence of forces external to the system, this is conservation of momentum
+ Suppose the external forces are due to uniform gravity
$$\vec{F}_{i,ext} = m_i \vec{g} \Rightarrow \dot{\vec{P}} = M\vec{g}$$
The CM moves like a single particle in gravity.

- Angular Momentum
  - Define $\vec{r}_i = \vec{R} + \vec{r}_i^*$ where $\vec{r}_i^*$ = position relative to CM, ie, position in CM frame
  - Then we re-write $\vec{J}$

+ $\vec{J} = \sum \left[m_i \vec{R}\times\dot{\vec{R}} + m_i \vec{r}_i^* \times \dot{\vec{R}} + m_i \vec{R}\times\dot{\vec{r}}_i^* + m_i \vec{r}_i^* \times \dot{\vec{r}}_i^* \right]$

+ But $\sum m_i \vec{r}_i^*$ = CM position in CM frame $= 0$

so
$$\vec{J} = M\vec{R}\times\dot{\vec{R}} + \vec{J}^*$$
= "orbital angular momentum" of CM + angular momentum around the CM ($\vec{J}^*$)

- Consider $\dot{\vec{J}} = \sum m_i \vec{r}_i \times \ddot{\vec{r}}_i$
  + From Newton's law
$$\dot{\vec{J}} = \sum_i \left(\vec{r}_i \times \vec{F}_{i,ext} + \sum_j \vec{r}_i \times \vec{F}_{ij} \right)$$
  where $\vec{F}_{ij}$ = force on i from j
  + But
$$\sum_{i,j} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\sum (\vec{r}_i - \vec{r}_j)\times\vec{F}_{ij} \quad \text{by } 3^{rd} \text{ law}$$

  if $\vec{F}_{ij}$ is central, this vanishes
  + In this case
$$\dot{\vec{J}} = \vec{\tau}_{ext} \quad \text{(external torque)}$$
  + But in fact we know from Noether's theorem that this must be true even for noncentral forces as long as the internal potential is rotationally symmetric. This now includes magnetic forces.

- What if the CM is accelerated?
  + The derivative of the "orbital" angular momentum is
$$M\vec{R}\times\ddot{\vec{R}} = \vec{R}\times\vec{F}_{ext}$$
  + Therefore,
$$\dot{\vec{J}}^* = \dot{\vec{J}} - \vec{R}\times\vec{F}_{ext} = \sum_i(\vec{r}_i-\vec{R})\times\vec{F}_{i,ext} = \sum \vec{r}_i^{\,*}\times\vec{F}_{i,ext}$$
  + This is the torque around the CM position generated by physical and not fictitious forces even if the CM is accelerating. This works out b/c $\vec{R}^*=0$

— Energy
- Kinetic energy also splits into CM + CM frame parts
$$T = \tfrac{1}{2}\sum m_i\dot{\vec{r}}_i^{\,2} = \tfrac{1}{2}M\dot{\vec{R}}^2 + T^*, \quad T^* = \tfrac{1}{2}\sum m\dot{\vec{r}}_i^{\,*2}$$
- The time derivative is
$$\dot{T} = \sum_i \dot{\vec{r}}_i\cdot\vec{F}_{i,ext} + \sum_{i,j}\dot{\vec{r}}_i\cdot\vec{F}_{ij}$$
  + The internal forces are $2\sum_{i<j}(\dot{\vec{r}}_i-\dot{\vec{r}}_j)\cdot\vec{F}_{ij}$
  + In a rigid body, $|\vec{r}_i-\vec{r}_j|^2 = \text{const} \Rightarrow (\vec{r}_i-\vec{r}_j)\cdot(\dot{\vec{r}}_i-\dot{\vec{r}}_j)=0$ so a central internal force does no work.
  + If $\vec{F}_{ij}$ is conservative, relativity principle means it is central and arises from $V_{ij}(|\vec{r}_i-\vec{r}_j|)$
    We can define an internal potential
$$V_{int} = \sum V_{ij}$$
    and
$$\frac{d}{dt}(T+V_{int}) = \sum\dot{\vec{r}}_i\cdot\vec{F}_{i,ext}$$
  + If external forces are conservative, there is a conserved total energy $T+V_{int}+V_{ext}$
- CM frame energy
  + Even if the CM is accelerated, we can see
$$\frac{d}{dt}(T^*+V_{int}) = \sum\dot{\vec{r}}_i^{\,*}\cdot\vec{F}_{i,ext}$$
    for physical external forces. Again, fictitious forces don't contribute.

+ For motion in uniform gravity,

$$T = \tfrac{1}{2} M \dot{\vec{R}}^2 + T^*, \qquad V = V_{int} - \sum m_i \vec{r}_i \cdot \vec{g} = V_{int} - M\vec{R}\cdot\vec{g}$$

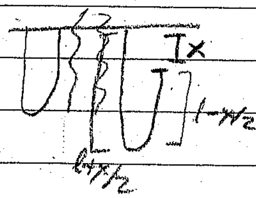
$$\Rightarrow \quad L = \left(\tfrac{1}{2} M \dot{\vec{R}}^2 + M\vec{g}\cdot\vec{R}\right) + \left(T^* - V_{int}\right)$$

so motion separates into motion of CM and motion around CM (as we've argued before). This means there is no net external torque also. Direct calculation is

$$\vec{\tau}_{ext} = \sum \vec{r}_i^{\,*} \times \vec{F}_{i,ext} = \left(\sum m_i \vec{r}_i^{\,*}\right) \times \vec{g} = 0.$$

— Examples: "Floppy Body" Motion

A few cases where we can use these ideas for motion of non-rigid objects



- Hanging String: A string of length $2l$ and linear mass density μ is attached at one end to the ceiling. The other end is initially at the ceiling, then falls.

+ Suppose the right-hand end has fallen distance x. If we ignore any horizontal swinging, we just need to understand the CM motion. The CM position is

$$X = \left[m_{left}\,(l + x/2)/2 + m_{right}\left(x + (l - x/2)/2\right)\right]/2\mu l$$

with $m_{left} = \mu(l + x/2)$, $m_{right} = \mu(l - x/2)$

$$\Rightarrow \quad X = (l + x - x^2/4l)/2$$

+ The total momentum (downward) is therefore

$$\vec{P} = (2\mu l)\dot{X} = \mu(l - x/2)\dot{x}$$

+ External forces are gravity and tension on the left end, so

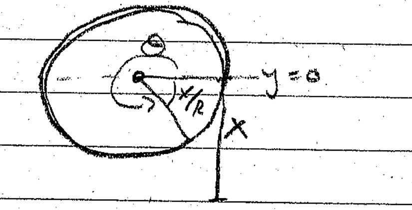
$$\dot{\vec{P}} = \mu(l - \tfrac{x}{2})\ddot{x} - \tfrac{\mu}{2}\dot{x}^2 = (2\mu l)g - T$$

If we assume that there is no tension at the free end, x is in free fall, so $\ddot{x} = g$, $\dot{x} = \sqrt{2gx}$. Then

$$T = \mu l g + 3\mu g x/2$$

+ There are actually a number of ways a falling chain behaves differently — horizontal motion is important.

- Unwinding Rope: A rope of linear density μ wraps once around a disk or cylinder of radius R. The disk can rotate around its center to unwind the rope



  + The vertical center of mass of the rope is the mass-weighted avg of the vertical segment CM + the wound segment CM

$$Y = \frac{1}{2\pi R\mu}\left\{ (\mu x)\left(-x/2\right) + \mu R^2 \int_0^{2\pi - x/R} d\theta \, \sin\theta \right\}$$

so potential energy

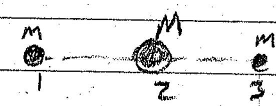
$$V = -MYg = g\left[x^2/2R + \cos(x/R) - 1\right]$$

  * The kinetic energy is due to the speed of the rope, which has speed $\dot{x}$ all along, and the rotation of the disk with $\omega = \dot{x}/R$. So

$$T = \tfrac{1}{2}\mu(2\pi R)\dot{x}^2 + \tfrac{1}{2} I \dot{x}^2/R^2$$

  + Conservation of energy gives speed as function of position, etc

- Linear Triatomic Molecule



Take a molecule w/ 3 atoms in a line (in equilibrium). The middle has mass M, other 2 mass m. Want to describe types of motion

  + First, we can work in CM frame. There can be rotational motion (ie, rigid body) around CM and also small vibrations

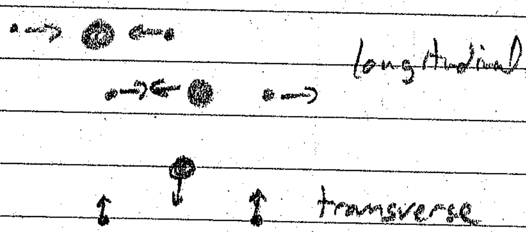
  + In CM frame, $\vec{r}_2 = -(m/M)(\vec{r}_1 + \vec{r}_3)$
  Purely vibrational motion has $\vec{J} = 0$.

  + The remaining vibrational motion has normal modes (in the xy plane):

$$x_1 = -x_3, \quad x_2 = 0$$
$$x_1 = x_3, \quad x_2 = -2m x_1/M$$

and

$$y_1 = y_3, \quad y_2 = -2m y_1/M$$



- Two Particles
  - There is a further simplification when there are only 2 objects.
  - The CM frame kinetic energy is
    $$T^* = \frac{1}{2} m_1 \dot{\vec{r}}_1^{*2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{*2}$$
    - Define the relative position $\vec{r} = \vec{r}_1^* - \vec{r}_2^*$
      This is actually also $\vec{r} = \vec{r}_1 - \vec{r}_2$ for any frame related by shift of origin or velocity
    - We also know $m_1 \vec{r}_1^* + m_2 \vec{r}_2^* = 0$, so
      $$\vec{r}_1^* = m_2 \vec{r}/M, \quad \vec{r}_2^* = -m_1 \vec{r}/M$$
    - Simplifying
      $$T^* = \frac{1}{2}\left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\right) \dot{\vec{r}}^2 = \frac{1}{2} \mu \dot{\vec{r}}^2$$
      where the reduced mass is
      $$\mu = m_1 m_2 / M \quad (\text{and is always } \mu < m_1, m_2)$$
    - This is like a single particle b/c the motion of the 2^{nd} particle "mirrors" the 1^{st}.
  - Also, consider the angular momentum
    - $\vec{J}^* = m_1 \vec{r}_1^* \times \dot{\vec{r}}_1^* + m_2 \vec{r}_2^* \times \dot{\vec{r}}_2^* = \mu \vec{r} \times \dot{\vec{r}}$
    - Again, we have the angular momentum of a single effective particle of mass μ following the path $\vec{r}$.
  - We will look at consequences + applications next.