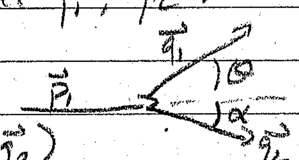
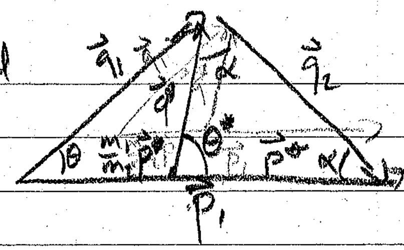
- Applications
  - Two-Body Collisions Dynamics: CM vs
    We want to understand the relation between the CM frame + lab frame descriptions of a collision
    - We will consider elastic collisions for simplicity
    - CM frame description
      + Elastic because kinetic energy is conserved. That means (1) no energy loss to heat (2) forces between objects $\to 0$ at large distance ($V \to$ constant)
      + The kinetic energy is (in terms of relative position)
        $$T = \tfrac{1}{2}\mu \dot{\vec{r}}^{\,2} \equiv \vec{p}^{*2}/2\mu, \quad p^* \equiv \mu\dot{\vec{r}} = m_1\dot{\vec{r}}_1^* = -m_2\dot{\vec{r}}_2^*$$
        Call $\vec{p}^*$ the CM frame momentum (of 1 object)
      + If $\vec{p}^*$ and $\vec{q}^*$ are initial + final CM frame momenta,
        $$|\vec{p}^*| = |\vec{q}^*|$$ no matter the forces
      + The objects change direction only and not speed in CM frame
      + The scattering angle θ^* is the deflection of each object and is the angle between $\vec{p}^*$ and $\vec{q}^*$
    - The (fixed target) lab frame
      + This is a common experiment design where object 2 is initially at rest
      + Define initial $\vec{p}_1$, $\vec{p}_2 = 0$ and final $\vec{q}_1$, $\vec{q}_2$ momenta
      + The 1st particle has scattering angle θ (angle between $\vec{p}_1$ + $\vec{q}_1$) while the 2nd has recoil angle α (between $\vec{p}_1$ + $\vec{q}_2$)



      + It's often easier to work in CM frame + convert
    - Converting frames.
      + We know that initially
        $$\vec{p}_1 = m_1\dot{\vec{r}}_1 = M\dot{\vec{R}} \quad \text{and} \quad 0 = \dot{\vec{r}}_2 = \dot{\vec{R}} + \dot{\vec{r}}_2^*$$
        $$\Rightarrow \vec{p}_1 = (M/m_2)\vec{p}^*, \quad \dot{\vec{R}} = \vec{p}^*/m_2$$
      + After the collision,
        $$\vec{q}_1 = m_1(\dot{\vec{R}} + \dot{\vec{r}}_1^*) = (m_1/m_2)\vec{p}^* + \vec{q}^* \quad \text{and}$$
        $$\vec{q}_2 = m_2(\dot{\vec{R}} + \dot{\vec{r}}_2^*) = p^* - q^*$$



+ We can therefore construct nested triangles of the momenta. The scattering + recoil angles sit at some vertices

+ The recoil angle appears at 2 vertices b/c $|\vec{p}^*| = |\vec{q}^*|$
This tells us that $\alpha = \frac{1}{2}(\pi - \theta^*)$ and
$|\vec{q}_2| = 2|\vec{p}^*| \sin(\frac{1}{2}\theta^*)$ (by dropping a $\perp$)

+ The fractional energy transfer in lab frame is the KE of the final 2nd particle relative to the total, or

$$(|\vec{q}_2|^2/2m_2)/(|\vec{p}_1|^2/2m_1) = \frac{(2|\vec{p}^*|^2 \sin^2(\theta^*/2)/m_2)}{M^2|\vec{p}^*|^2/2m_1 m_2^2}$$

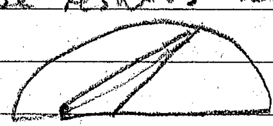
$$= (4m_1 m_2/M^2)\sin^2(\theta^*/2)$$

The max occurs when $\theta^* = \pi$, particles reversed in CM frame. It only $\to 1$ when $m_1 = m_2$ (Newton's cradle)

+ Meanwhile, dropping a $\perp$ from the top vertex relates the scattering angles

$$\tan\theta = \sin\theta^* / (\cos\theta^* + m_1/m_2)$$

+ Max lab scattering angle:
If $m_1 < m_2$, $\theta \to \pi/2$ when $\cos\theta^* = -m_1/m_2$, then increases to π
But if $m_1 > m_2$, we can never have $\theta = \pi/2$. Instead, θ increases to θ_{max}, then decreases back to 0. With a little work, we see $\sin\theta_{max} = m_2/m_1$
These results are encapsulated in the diagrams



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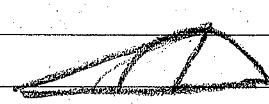

— Cross Sections in Different Frames

- Review of definitions
  + We imagined a flux f of incoming particles (ie, #/time) hitting a target of cross sectional area σ, so #collisions/time $= f\sigma$.

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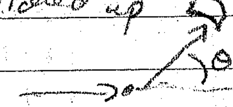
  + This lets us define generally $\sigma \equiv$ (collisions/time)$/f$.

+ The number of scattering events that are picked up by a detector of area dA at distance R from the target is



$$\text{scatterings/time} = f\left(\frac{d\sigma}{d\Omega}\right) dA/R^2$$

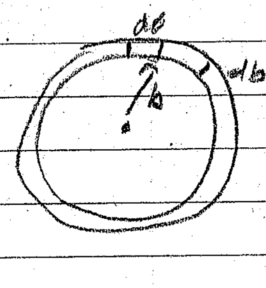
+ This defines differential cross section $d\sigma/d\Omega$

If we think of the sphere of radius R in terms of polar angles, the solid angle of dA is $d\Omega = dA/R^2 = \sin\theta \, d\theta \, d\phi$

+ The polar angle θ is the scattering angle, which is determined by impact parameter b, typically monotonically decreasing.

in CM frame + in lab frame for $m_1 < m_2$

Then $d\sigma = b|db| \, d\phi = (d\sigma/d\Omega) \, d\Omega$.



- Lab vs CM frame Cross Sections

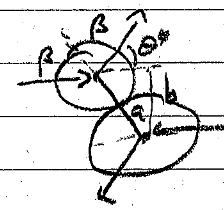
+ Take a lab frame with target particles initially at rest. The total targets is $n_2 V$, where n_2 = number density, V = volume, and total flux is $n_1|\vec{v}|$, where n_1 = number density + $\vec{v}$ = velocity of initial incoming particles. So total rate of scatterings into dA is
$$dw = n_1 n_2 |\vec{v}| V (d\sigma/d\Omega) \, dA/R^2$$

+ But in CM frame, the particles move past each other at a rate given by the relative velocity $\Delta\vec{v} = \vec{v}_1 - \vec{v}_2$. The scattering rate is
$$dw = n_1 n_2 |\Delta\vec{v}| V (d\sigma/d\Omega^*) \, dA/R^2$$
but the area is oriented differently

+ It is often easier to get the lab cross section by calculating the CM cross section and converting
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega^*}\left(\frac{\sin\theta^* \, d\theta^*}{\sin\theta \, d\theta}\right) = \frac{d\sigma}{d\Omega^*} \frac{d\cos(\theta^*)}{d\cos(\theta)}$$
We can get the conversion factor from
$$\tan\theta = \sin\theta^* / (\cos\theta^* + m_1/m_2)$$
A simple case is $m_1 = m_2 \Rightarrow \theta^* = 2\theta$, so
$$\frac{d\cos\theta^*}{d\cos\theta} = \frac{d}{d\cos\theta}(2\cos^2\theta - 1) = 4\cos\theta.$$

+ Some complication in this conversion for $m_1 > m_2$ b/c of θ_{max}

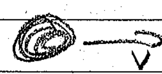
- Example: Hard Sphere Scattering
  + Take 2 spheres of radii a_1 + a_2, masses m_1 + m_2. They have an elastic contact interaction, so they scatter whenever the impact parameter $b \leq a \equiv a_1 + a_2$. The total cross section is $\sigma = \pi a^2$
  + Differential cross section determined by realizing scattering is reflection from contact plane. As per figure, this means $\theta^* = \pi - 2\beta$

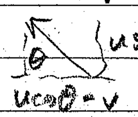


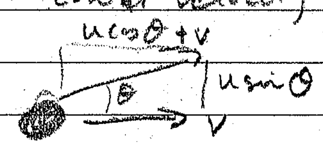
  + Then we have
$$b = a \sin\beta = a\cos(\theta^*/2)$$
so: $$b|db| = (a^2/2)\cos(\theta^*/2)\sin(\theta^*/2)\,d\theta^* = (a^2/4)\sin\theta^*\,d\theta^*$$
and
$$d\sigma/d\Omega^* = a^2/4.$$ Same as when $m_2 \to \infty$ from the fall.
  + But what about lab frame with m_2 at rest initially? Take $m_1 = m_2$ for simplicity. Then $\theta < \pi/2$ — no backwards scattering. With our conversion
$$d\sigma/d\Omega = a^2\cos\theta.$$
This also integrates to give $\sigma = \pi a^2$.

— Orbital Mechanics

- Gravity Assist: A space probe approaches a planet with relative speed u.
  + Since the planet is effectively infinitely massive, CM frame is planet rest frame. Probe leaves w/ relative speed u.
  + In solar frame, the probe initial "horizontal" velocity is $(u\cos\theta - v)$ for planetary velocity v.







After, horizontal velocity of probe is $u\cos\theta + v$.
Increase of twice planetary orbit speed!

- Elliptical orbits
  + The actual Lagrangian is
$$L = \tfrac{1}{2}\mu\dot{r}^2 + GM\mu/r \quad \text{since } m_1 m_2 = \mu M.$$

+ Therefore, the orbit is the same as an object of mass μ orbiting a fixed center of mass M = total mass.

+ In our solar system, we almost always have $m_2 >> m_1$, so $M \to m_2$, $\mu \to m_1$. Earth-moon is a slight exception

+ There is a slight change to Kepler's 3rd law:

$$(T/2\pi)^2 = a^3/GM$$, where a = semi-major axis for relative position and M = total mass.

This changes slightly for each planet (a moon)

+ Each orbiting object orbits the center of mass in an ellipse with $\vec{r}_1^* = m_2\vec{r}/M$, $\vec{r}_2^* = -m_1\vec{r}/M$.

That means the semi-major axes are

$$a_1 = m_2 a/M, \quad a_2 = m_1 a/M.$$

• Tidal Friction

+ Think of the high tides as bulges of water. These want to rotate w/ the earth, but the tidal forces of the moon pull them back toward the moon. This tends to slow down the rotation

+ Quantitatively, the CM frame angular momentum is

$$\vec{J}^* = \mu \vec{r} \times \dot{\vec{r}} + \vec{J}_\oplus^* + \vec{J}_m^*$$

where $\vec{J}_\oplus$ + $\vec{J}_m$ are due to earth's + moon's rotations, $\vec{J}_m$ is smaller than $\vec{J}_\oplus$ and the orbital term

+ Angular momentum conservation is

$$J^* = \mu a^2 \Omega + I_\oplus \omega$$

where a = orbit semimajor axis, Ω = orbit frequency, $I_\oplus$ = earth moment of inertia, ω = rotation frequency of earth.

+ But we further know

$$\Omega^2 a^3 = GM \qquad \text{(Kepler's 3rd law)}$$

so

$$J^* = \mu\sqrt{GMa} + I_\oplus \omega$$

+ So, as the tidal friction slow earth's rotation, the orbit semi-major axis must decrease. This is a transfer of energy from earth to orbit.

• Restricted 3-body Problem

+ In general, it's impossible to solve for the motion of 3 objects interacting gravitationally except on a computer. This is the "3-body problem". This might describe the sun, earth, & moon, a planet around binary stars, etc. The more general N-body problem describes a solar system and (importantly) the formation of structure in the universe.

+ In the 3-body problem, call the objects the primary, secondary, and tertiary in order of most to least massive ($m_1 > m_2 > m_3$)

+ To make progress, we will look at the "restricted 3-body problem":
  1) We assume the tertiary is very light $m_3 \ll m_1, m_2$ so its gravity does not meaningfully affect primary or secondary
  2) All motion is in one plane
  3) The primary & secondary orbit their CM circularly

+ Work in a frame rotating with the primary & secondary and their CM at the origin. They are located at $a_1 = m_2 a/(m_1+m_2)$ and $-a_2 = -m_1 a/(m_1+m_2)$ on the $\hat{x}$ axis. Angular velocity of the frame is $\vec{\omega} = \omega\hat{z}$ with $\omega^2 = G(m_1+m_2)/a^3$.

+ If the tertiary is at $\vec{r} = x\hat{x} + y\hat{y}$ in the rotating frame, it experiences force (including fictitious forces)

$$\vec{F} = -m\left(\frac{Gm_1}{r_1^3}\vec{r}_1 + \frac{Gm_2}{r_2^3}\vec{r}_2\right) - 2m\vec{\omega}\times\dot{\vec{r}} + m\omega^2\vec{r}$$

where

$\vec{r}_1 = \vec{r} - a_1\hat{x}$, $\vec{r}_2 = \vec{r} + a_2\hat{x}$ are the relative positions

+ There are 5 Lagrangian points where $\vec{F}$ vanishes on a stationary tertiary. L_1 to L_3 are on the x axis. L_1 is between m_1 & m_2, L_2 is past m_2, & L_3 is past m_1. These are unstable. L_4 & L_5 are the vertices of equilateral triangles with primary & secondary at the others. Due to Coriolis force, there are stable small motions near L_4 & L_5.