Hamiltonian Mechanics + Symmetries

- Hamilton's Equations
  - Changing Variables
    + The Lagrangian is a function $L = L(q_1, \ldots q_N, \dot{q}_1, \ldots \dot{q}_N)$ of coordinates + time derivatives
    + While we think of varying the action wrt q, the E-L eqns treat q + $\dot{q}$ as $2N$ indep variables
    + We can swap out the $\dot{q}$ dependence for the canonical momenta $p_i = \partial L / \partial \dot{q}_i$ by inverting the relation to $\dot{q}_i \to \dot{q}_i(q, p)$

  - To change the independent variables of a function, we can use a __Legendre transform__
    + If $f = f(x)$, define $y = df/dx\,(x)$ and invert.
    + Then define
      $$g(y) = xy - f(x(y))$$
    + The reciprocal relationship is $dg/dy = x(y)$

      Check:
      $$dg/dy = x + y\frac{dx}{dy} - \frac{df}{dx}\frac{dx}{dy} = x \quad ✓$$
    + We define the __Hamiltonian__ $H(q,p)$ as the Legendre transform of the Lagrangian $L(q, \dot{q})$

      $$H(q,p) = \left(\sum_i p_i \dot{q}_i\right) - L(q, \dot{q}(q,p))$$
    + Other Legendre transform pairs in physics are mostly in thermodynamics

      $$\text{energy} \xleftrightarrow{P,V} \text{enthalpy} \xleftrightarrow{S,T} \text{Gibbs free energy}$$
      $$\text{energy} \xrightarrow[S,T]{} \text{Helmholtz free energy} \xrightarrow{P,V} \text{Gibbs free energy}$$

  - Hamilton's eqns follow from E-L eqns

    + The 1st is from the canonical momentum + Legendre transform
      $$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} - \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i$$

+ The second is

$$\frac{\partial H}{\partial q_i} = \sum_j p_j \frac{\partial \dot q_j}{\partial q_i} - \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial L}{\partial \dot q_j}\frac{\partial \dot q_j}{\partial q_i}$$

$$= -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = -\dot p_i$$

+ Put together,

$$\dot q_i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q_i}$$

are Hamilton's eqns

+ We have swapped N 2nd order DEs for $2N$ 1st order. This can be advantageous, esp. for computing

- Meaning of the Hamiltonian
  + Suppose we have a natural system, so Cartesian coords $x_i = x_i(q)$ with no explicit time dependence
  + Then the usual kinetic energy is given by $\dot x_i = \sum_j (\partial x_i/\partial q_j)\dot q_j$ to be

$$T = \sum_{i,j} A_{ij}(q)\,\dot q_i \dot q_j, \quad \text{2nd order homogeneous fct of } \dot q$$

  + From this, we see $\sum_i (\partial T/\partial \dot q_i)\dot q_i = 2T$
  + With a conservative force $V = V(q)$, then

$$H = \sum_i p_i \dot q_i - L = \left(\sum_i \frac{\partial T}{\partial \dot q_i}\dot q_i\right) - (T - V)$$
$$= T + V$$

This is total energy!

  + What's the time derivative?

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \frac{\partial H}{\partial q_i}\dot q_i + \sum_i \frac{\partial H}{\partial p_i}\dot p_i = \frac{\partial H}{\partial t} \text{ by Hamilton's eqns}$$

That means the Hamiltonian is conserved as long as it does not have explicit time dependence

- One note: In the Lagrangian formalism, q + $\dot q$ and the variations δq and $\delta \dot q$ are related by differentiation — NOT indep. But in Hamiltonian mechanics, q + p are treated as indep variables. It is possible to transform coords such as $q \to \tilde q(q,p)$, $p \to \tilde p(q,p)$

– Examples

• Spherical Pendulum (Again)

+ Recall

$$L = \tfrac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) + mgl\cos\theta + \text{const.}$$

+ The canonical momenta are

$$p_\theta = ml^2\dot{\theta}, \quad p_\phi = ml^2\sin^2\theta\,\dot{\phi}$$

with Hamiltonian

$$H = p_\theta^2/ml^2 + p_\phi^2/ml^2\sin^2\theta - \tfrac{1}{2}\left(\frac{p_\theta^2}{ml^2} + \frac{p_\phi^2}{ml^2\sin^2\theta}\right) - mgl\cos\theta + \text{const}$$
$$= p_\theta^2/2ml^2 + p_\phi^2/2ml^2\sin^2\theta - mgl\cos\theta + \text{const}$$

+ Since H does not depend on ϕ, $\dot{p}_\phi = 0$

Because H depends on p_ϕ (not $\dot{\phi}$), which is constant, you see the effective potential

$$H = \frac{p_\theta^2}{2ml^2} + U(\theta), \quad U = \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta + \text{const}$$

(Remember that this didn't work in the Lagrangian version)

• Electromagnetism

+ Remember that we had a velocity dependent potential, or

$$L = \tfrac{1}{2} m\dot{\vec{x}}^2 + q\dot{\vec{x}}\cdot\vec{A} - q\Phi$$

+ The canonical momenta are

$$\vec{p} = m\dot{\vec{x}} + q\vec{A}$$

so

$$H = \vec{p}\cdot(\vec{p} - q\vec{A})/m - \tfrac{1}{2}(\vec{p} - q\vec{A})^2/m - q\vec{A}\cdot(\vec{p} - q\vec{A})/m + q\Phi$$

$$= \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\Phi$$

+ Note that the canonical momentum is not the usual one. The Hamiltonian is also not the total energy

– Symmetries + Conservation Laws

• Ignorable / Cyclic Coordinates

+ In our spherical pendulum example, $\partial H/\partial\phi = 0$, so p_ϕ was conserved.

+ Coordinates on which H does not depend are called *ignorable* or *cyclic* and lead to conservation laws and the canonical momenta enter into effective potentials

+ There are advanced techniques based on finding cyclic coordinates (this will wait!)

+ The conservation laws are related to a symmetry of motion, ie, translational symmetry (linear momentum), axial symmetry (J_z), spherical symmetry ($\vec{J}$)

+ But what if there is a symmetry but we have no cyclic coord? For ex, rotational symmetry w/ Cartesian coords.

- Transformations + the Poisson Bracket
  + In Hamiltonian mechanics, an infinitesimal transformation changes $q \to q + \delta q$, $p \to p + \delta p$, with $\delta q, \delta p \propto \delta\lambda$ a small parameter. Ex for a rotation, $\delta\lambda = \delta\theta$ = small angle of rotation
  + A transformation is *generated* by function $G(q,p,t)$ if

$$\delta q_i = \frac{\partial G}{\partial p_i}\delta\lambda, \quad \delta p_i = -\frac{\partial G}{\partial q_i}\delta\lambda$$

  + Hamilton's eqns tell us H generates translations of time b/c $\delta q_i = \dot{q}_i \delta t$, $\delta p_i = \dot{p}_i \delta t$
  + The change of general quantity $F(q,p,t)$ is

$$\delta F = \sum_i \left(\frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_i}\delta p_i\right) = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\delta\lambda$$

This is the *Poisson bracket*

$$\{F,G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

(It is related to the quantum commutator)

  + Note that $\{F,G\} = -\{G,F\}$

- Time dependence + Conservation
  + We noted that H generates time translations based on δq, δp.
  + A general quantity $F(q,p,t)$ has

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F}{\partial p_i}\dot{p}_i = \frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \frac{\partial F}{\partial t} + \{F, H\} \quad \text{by Hamilton's eqns}$$

+ Now suppose we have a transformation generated by $G(q,p)$ (ie, $\partial G/\partial t = 0$). It is a symmetry of H if

$$\delta H = \{H, G\}\,\delta\lambda = 0$$

and it is conserved if

$$dG/dt = \{G, H\} = 0$$

Because the P.B. is antisymmetric, the generator of a symmetry transformation is conserved.

This is Noether's theorem (Emmy Noether)

+ There is a generalization when $\partial G/\partial t \neq 0$. See reading

• Lagrangian version of Noether's theorem

+ Under a transformation $q \to q + \delta q$, the Lagrangian changes

$$\delta L = \sum_i \left(\partial L/\partial q_i\, \delta q_i + \partial L/\partial \dot{q}_i\, d\delta q_i/dt\right)$$

If this is a symmetry, $\delta L = 0$ no matter the trajectory $q(t)$.

+ On the other hand, if the trajectory is the classical one, it satisfies the E-L eqns

$$\partial L/\partial q_i = \frac{d}{dt}\left(\partial L/\partial \dot{q}_i\right)$$

+ Then, for a symmetry, classical trajectories satisfy

$$0 = \delta L = \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right)$$

$$= \frac{dG}{dt} \quad \text{where} \quad G = \sum \frac{\partial L}{\partial \dot{q}_i}\delta q_i$$

+ This is the original version + tells you how to find the conserved quantity

+ What matters is actually the action $\delta S = 0$, so we can have $\delta L = dg/dt$ for some function g.

This leads to a more general version (as above)

• Example: rotation around z axis (see reading for more)

+ The transformation is (for each particle)

$$\delta x = -y\,\delta\phi, \quad \delta y = x\,\delta\phi, \quad \delta z = 0$$
$$\delta p_x = -p_y\,\delta\phi, \quad \delta p_y = p_x\,\delta\phi, \quad \delta p_z = 0$$

+ The generator is
$$J_z = \sum (x p_y - y p_x)$$
Then
$$\delta H = \sum\left(-y\frac{\partial H}{\partial x} + x\frac{\partial H}{\partial y} - p_y\frac{\partial H}{\partial p_x} + p_x\frac{\partial H}{\partial p_y}\right)$$
If we include all components, this vanishes if all $\vec{x}, \vec{p}$ dependence is through dot products

- Other theoretical developments
  - Liouville's Theorem
    + In Hamiltonian mechanics, we describe the state of a system by position in phase space, which is the 2N-dimensional space given by (q_i, p_i)

    + In stat. mech, we might ask about the evolution in phase space of an ensemble of systems w/ similar initial (q,p) values. We can describe them via a density in phase space

    + How does this density change in time? Consider any kind of density in space. If we integrate over a region of space
$$\frac{d}{dt}\left(\int_V d^n x\,\rho\right) = -\oint_A d\vec{s}\cdot(\rho\vec{v})$$
where $\vec{v}$ is the velocity of the stuff making up the density. Gauss's theorem lets us write
$$\partial\rho/\partial t = -\vec{\nabla}\cdot(\rho\vec{v}) \qquad \text{Continuity eqn}$$

    + In phase space, the continuity equation is
$$\frac{\partial\rho}{\partial t} = -\sum_i\left[\frac{\partial}{\partial q_i}(\rho\dot{q}_i) + \frac{\partial}{\partial p_i}(\rho\dot{p}_i)\right]$$
$$= -\sum_i\left[\frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\rho}{\partial p_i}\dot{p}_i\right] - \rho\sum_i\left[\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i}\right]$$

+ The latter 2 terms are
$$\partial^2 H/\partial q_i \partial p_i - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0 \quad \text{by Hamilton's eqns}$$

+ Then
$$\partial\rho/\partial t + \sum_i \left[\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right] = \frac{d\rho}{dt} = 0$$

This is Liouville's theorem: the density of similar systems in phase space is unchanged following the paths of these systems in time

+ Useful for thermal systems, paths of particles in accelerators, stars in galaxies, etc

• The virial theorem

+ Suppose we have a set of particles such that $\vec{x}$ and $\vec{p}$ stay finite for all time. Consider $F = \vec{x} \cdot \vec{p}$

+ Then the time average
$$\langle dF/dt \rangle = \frac{1}{T}\int_0^T dt \frac{dF}{dt} = \frac{1}{T}\left(F(T) - F(0)\right) \to 0$$
at long times T

+ But also $dF/dt = \sum_{\text{particles}} (\dot{\vec{x}} \cdot \vec{p} + \vec{x} \cdot \dot{\vec{p}})$
$$\Rightarrow \langle \sum \dot{\vec{x}} \cdot \vec{p} \rangle = -\langle \sum \vec{x} \cdot \dot{\vec{p}} \rangle$$

+ For the usual kinetic energy, this is
$$2\langle T \rangle = -\langle \sum \vec{F} \cdot \vec{x} \rangle \qquad \text{virial theorem}$$

+ You can use this to derive the ideal gas law, for ex.

+ If $\vec{F} = -\vec{\nabla} V$ with $V \propto r^{n+1}$ $\langle T \rangle = \left(\frac{n+1}{2}\right) \langle V \rangle$

This is useful for harmonic oscillators, gravity, etc.