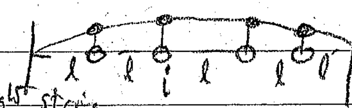
Coupled Oscillators + Waves

- Beads on a string:



- Consider beads of mass m on light string
  + that are at equilibrium for a straight string with beads separated by distance l.
  + Ignore gravity as negligible compared to string tension
  + The beads will act as coupled oscillators

- Tension + Energy
  + Tension F is the force exerted by the microstructure of the string against extending the string.
  + The potential energy is therefore

$$V = \int F\,dl \approx F_0\,\delta l + \tfrac{1}{2}F_1\,\delta l^2 + \cdots$$

where

$$F(l+\delta l) = F_0 + F_1\,\delta l + \cdots \qquad \text{(Taylor expansion)}$$

  + The length of the string segment from bead i to $(i+1)$ is

$$l + \delta l = \sqrt{(l + x_{i+1} - x_i)^2 + (y_{i+1}-y_i)^2}$$

$$\approx l + (x_{i+1}-x_i) + \tfrac{1}{2l}(x_{i+1}-x_i)^2 + \tfrac{1}{2l}(y_{i+1}-y_i)^2 + \cdots$$

  + where x_i, y_i are longitudinal + transverse displacements of bead i from equilibrium
  + If there are N beads, set $x_0 = x_{N+1} = y_0 = y_{N+1} = 0$ for the attachments to the walls. Then

$$V = \sum_{i=0}^{N} F_0 (x_{i+1}-x_i) + \frac{1}{2}\sum_{i=0}^{N}\Big[(F_1 + F_0/l)(x_{i+1}-x_i)^2 + (F_0/l)(y_{i+1}-y_i)^2\Big]$$

$$= \frac{1}{2}\kappa \sum_i (x_{i+1}-x_i)^2 + \frac{1}{2}(F_0/l)\sum_i (y_{i+1}-y_i)^2$$

(Note that the linear terms cancel between the i and $i+1$ terms in the sum as appropriate for equilibrium)

- Transverse Motion (for definiteness)
  + The Lagrangian is

$$L = \frac{1}{2}m\sum_i \dot{y}_i^2 - \frac{1}{2}\frac{F_0}{l}\sum_i (y_{i+1}-y_i)^2$$

Coupled oscillators!

+ The potential matrix is $\left[V_{ij}\right] = \frac{F_0}{l}\begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & 0 \cdots \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \\ & & & \end{bmatrix}$

Since kinetic term has $[m_{ij}] = m\mathbb{1}$, can just diagonalize V. That's easy for small N (we did $N=2$) but gets harder quickly

+ Instead, try a trick. The EOM are

$$m\ddot{y}_j - F_0/l\,(y_{j+1} - 2y_j + y_{j-1}) = 0 \quad \text{(use } j \text{ instead of } i \text{ for clarity)}$$

We will guess that the normal mode of frequency ω has $y_j = a\sin(j\gamma)$. This ensures $y_0 = 0$.

+ The EOM becomes

$$a\sin(j\gamma)\left[-m\omega^2 + 2F_0/l\,(1-\cos\gamma)\right] = 0$$

after using angle addition formula

$$\sin((j+1)\gamma) + \sin((j-1)\gamma) = 2\sin(j\gamma)\cos\gamma$$

+ $\Rightarrow$

$$\omega^2 = \frac{2F_0}{ml}(1-\cos\gamma) = \frac{4F_0}{ml}\sin^2(\gamma/2)$$

+ What are allowed values of γ? We need $\sin((N+1)\gamma) = 0$

so $\gamma_n = n\pi/(N+1)$, $n = 1, 2, 3, \ldots$.

But note that $n = N+1$ makes $y_j = a\sin(j\pi) = 0$

and in fact $\sin[(N+1+n)j\pi/(N+1)] = \sin(j\pi/(N+1))$

so $n = 1, 2, \ldots N$.

+ The normal modes are given by

$$y_j(t) = \sum_n \eta_n(t)\sin(nj\pi/(N+1))$$

Check that the Lagrangian diagonalizes!

+ This is the definition of phonons in solid state physics (quantized)

- Continuous Limit: model of massive string

  + Take the limit of ∞ beads but a fixed length L and mass of string:

$$m\to 0,\ N\to\infty,\ l\to 0 \text{ with } (N+1)l = L \text{ and } m/l = \mu$$

  + The position of a bead from the end of the string $x = jl$

  + Therefore,

$$y_j(t) = \sum_n \eta_n(t)\sin\left(\frac{nj\pi}{N+1}\right)$$

$$\rightarrow y_j(x,t) = \sum_n \eta_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

- Waves of a String

  - Energy + Lagrangian

    + Suppose a string has linear mass density μ and is held at equilibrium along the x axis between 2 points a distance L apart with tension F.

    + We'll study small transverse displacements $y(x,t)$. So keep only terms to 2nd order in y.

    + The kinetic energy of a small length dl of string is

$$dT = \tfrac{1}{2}(\mu\, dl)\, \dot{y}^2 = \tfrac{1}{2}\left(\mu\sqrt{1+(y')^2}\, dx\right)\dot{y}^2 \simeq \tfrac{1}{2}(\mu\, dx)\,\dot{y}^2$$

so KE is

$$T = \int_0^L dx\ \tfrac{1}{2}\mu\, \dot{y}^2$$

Note $\dot{y} = \partial y/\partial t$, $y' = \partial y/\partial x$

    + We've noted that the potential is given by the change in string length. This is

$$L + \delta L = \int_0^L dx\ \sqrt{1+(y')^2} \simeq L + \tfrac{1}{2}\int (y')^2$$

so

$$V = \int_0^L dx\ \tfrac{1}{2}F(y')^2$$

    + We can therefore write the action in terms of a <u>Lagrangian density</u> $\mathcal{L}$

$$S = \int dt \int_0^L dx\ \mathcal{L}(x,t;y,\dot{y},y')$$

with

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}F(y')^2$$

  - Least Action + EOM

    + The principle of least action tells us that classical behavior has $\delta S = 0$. We take

$$\delta S = \int dt \int dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\delta \dot{y} + \frac{\partial \mathcal{L}}{\partial y'}\delta y' \right)$$

    + Then we have

$$\delta\dot{y} = \frac{\partial}{\partial t}\delta y \quad \text{and} \quad \delta y' = \frac{\partial}{\partial x}\delta y$$

and $$\delta S = \int dt \int dx \left\{ \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) \right] \delta y + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \delta y\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'} \delta y\right) \right\}$$

+ The total time derivative term ends up being evaluated at initial and final times. But $\delta y(x, t_i) = \delta y(x, t_f) = 0$ b/c the variational problem always assumes the configuration $y(x)$ is fixed at the start and end times.
+ The total x derivative term is a bdy term evaluated at $x=0$ and $x=L$. But $y(x,t)$ satisfies Dirichlet b.c at those ends, so $\delta y(0,t) = \delta y(L,t) = 0$. So this vanishes also.
+ We are left with the E-L eqn
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0$$

- Wave Equation
  + The Lagrangian density for transverse motion of a string is
$$\mu \frac{\partial^2 y}{\partial t^2} - F \frac{\partial^2 y}{\partial x^2} = 0$$
  This is the <u>wave equation</u>. It can be written
  +
$$\ddot{y} = v^2 y''$$
  where the wave velocity is $v = \sqrt{F/\mu}$
  + The general solution is
$$y(x,t) = f(x+vt) + g(x-vt)$$
  where $f + g$ are generally independent functions.
  + Now take our Dirichlet b.c. $y(0,t) = y(L,t) = 0$.
  The 1st tells us
$$f(vt) = -g(-vt) \Rightarrow g(x-vt) = -f(vt-x)$$
  The 2nd says
$$f(vt+L) = f(vt-L) \Rightarrow f \text{ has period } 2L$$
  + To find $f(u)$, consider initial conditions
$$y(x,0) = f(x) - f(-x) \equiv f_0(x)$$
  In other words, the initial value of y determines the <u>odd</u> part of $f(u)$ on $0 < u < L$. We can also specify $\dot{y}(x,0) = vf'(x) - vf'(-x)$

even part

b/c the wave eqn is 2^{nd} order in time. For $f_+(u) = f(u) + f(-u)$, $f_+' = f'(u) - f'(-u)$ by the chain rule. So $\dot{y}$ sets the even part of f on $0<u<L$. Since $f_\pm$ are even/odd, we can extend them to $-L<u<L$ and therefore find all of f.

- Normal Modes / Standing Waves
  - + The wave eqn is separable, so we can take $y(x,t) = A(x)e^{i\omega t}$ to find the normal modes
  - + This gives

$$A'' = -(\omega^2/v^2)A$$

If we define $k = \omega/v$, $A(x) = A_k\cos(kx+\delta)$

  - + The b.c. $y(0,t) = y(L,t) = 0$ require $A(x) = A_n \sin(kx)$ with $k = n\pi/L$, $n = 1, 2, 3, \ldots$
  - + So the normal modes are standing waves such that an integral # of half-wavelengths fit on the string, oscillating at frequency $\omega_n = (n\pi/L)\sqrt{F/\mu}$

  - + We can also find the normal coordinates from the Lagrangian. The derivation (+ normal modes above) suggest we define

$$y(x,t) = \sum_n \eta_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

  - + Then

$$S = \int dt \int_0^L dx \left\{ \frac{1}{2}\mu \sum_{n,m} \dot{\eta}_n \dot{\eta}_m \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) - \frac{1}{2}F\sum_{n,m}\eta_n\eta_m\left(\frac{nm\pi^2}{L^2}\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\right\}$$

but

$$\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \int_0^L dx \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) = \frac{L}{2}\delta_{n,m}$$

so

$$S = \int dt \sum_n \left\{ \frac{1}{2}\mu\,\dot{\eta}_n^2 - \frac{1}{2}F\left(\frac{n\pi}{L}\right)^2\eta_n^2 \right\}$$

We see the appearance of the normal mode frequencies again

- Phase + Group Velocity
  - Phase Velocity
    + Let's ignore the boundary conditions and look only at the traveling wave sol'n $y(x,t) = f(x-vt)$
    Points of constant argument $u = x - vt$ clearly move right (positive x) at speed v.
    + Now let's ignore the EOM for a given wave. For example, the normal modes for beads on a string are standing waves but not from the wave eqn.
    We can consider terms in the wave of the form
    $$y(x,t) = A \exp[i(\omega t - kx)] \quad \text{for complex sol'n}$$
    + In this case, the position of constant phase $\phi = \omega t - kx$ moves at speed $v = \omega/k$ to the right. This is the phase velocity.
    + For the vibrating string, phase velocity is $\sqrt{F/\mu}$.
    But for the beads, the normal modes are given by $(j = x/l)$
    $$\exp[i(\omega t - n\pi x/l(N+1))] \Rightarrow k = n\pi/(N+1)l$$
    and
    $$\omega = 2\sqrt{\frac{F}{ml}} \sin(kl/2) \Rightarrow v = \sqrt{\frac{Fl}{m}} \frac{\sin(kl/2)}{kl/2}$$
    This is a nonlinear dispersion relation. It commonly appears for crystal vibrations in quantum.

  - Group velocity
    + Now suppose we have a more general "lump" of wave known as a wave packet
    $$y(x,t) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{i(\omega t - kx)} \quad \text{by Fourier transform}$$
    where $\omega = \omega(k)$
    + ~~Further~~ Then suppose that the amplitude function $A(k)$ is nonnegligible only for $k_0 - \Delta k < k < k_0 + \Delta k$, so take
    $$\omega(k) \approx \omega_0 + \omega_0'(k - k_0) + \cdots$$
    + The wave packet is therefore
    $$y(x,t) \approx \int dk\, A(k) \exp\{i[(\omega_0 - \omega_0' k_0)t + k(\omega_0' t - x)]\}$$

$$= e^{i(\omega_0 - \omega_0' k_0)t} \int dk\, A(k)\, e^{ik(\omega_0' t - x)} = e^{i(\omega_0 - \omega_0' k_0)t}\, y(x - \omega_0' t, 0)$$

+ This last form shows that the wave packet is approximately a fixed envelope moving at the <u>group velocity</u> $d\omega/dk$ to the right (and oscillating rapidly)

+ Note that the phase velocity is the wave velocity for an infinite sinusoidal wave for all time. It can't carry information. So the group velocity is the important one for comparison to light speed, for ex.