Time-Dependent Perturbation Theory

— The problem + formal solution

1) System has Hamiltonian $H = H_0 + H_1(t)$, where H_0 is time-indep. as usual and we know stationary states. Simple setup:

1) For $t<0$, $H_1(t)=0$. 2) H_1 "turns on" at $t=0$

3) H_1 "turns off" at $t=T$ $(H_1(t>T)=0)$.

- If we start in some stationary state $|\psi_1^0\rangle$, of H_0 at $t=0$,
  - what is the state at some later time (say $t=T$)?
  - Another way to phrase it: what is the probability that the system can be measured in a different stationary state $|\psi_2^0\rangle$ at $t=T$? (What is the probability of a transition?
- You never get transitions between stationary states except with time-dependence in H. This is how excited states decay, etc.
- Eigenstates of H_0 form a basis, so the full state is

$$|\Psi(t)\rangle = \sum_n c_n(t)\, e^{-iE_n^0 t/\hbar}\, |\psi_n^0\rangle$$ time-dep from H_0 only

with + normalization $\sum_n |c_n(t)|^2 = 1$

and + typical initial conditions $c_{\bar{n}}(0)=1$, $c_{n\neq\bar{n}}(0)=0$

i.e. start in state $|\Psi(t=0)\rangle = |\psi_{\bar{n}}^0\rangle$

- The Schrödinger eqn $i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$ becomes

$$\sum_n \left(i\hbar\,\dot{c}_n - c_n H_1\right)|\psi_n^0\rangle e^{-iE_n^0 t/\hbar} = 0$$

$$\Rightarrow \quad \dot{c}_m = \frac{-i}{\hbar}\sum_n \langle\psi_m^0|H_1|\psi_n^0\rangle c_n\, e^{-i(E_n^0-E_m^0)t/\hbar} \quad (*)$$ after inner product.

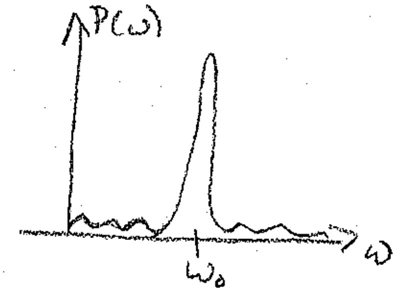
So far, this is exact.

- First-order Perturbation theory
  + $C_{\bar{n}}(t) = 1 + C^1_{\bar{n}}(t)$, $C_{n\neq\bar{n}}(t) = C^1_n(t) \Leftarrow$ ie, 1st order.
  + Then (#) becomes (b/c H_1 is 1st order, plug in C^0_n to RHS)
$$\dot{C}_{\bar{n}}(t) = \frac{-i}{\hbar}\langle \psi^0_{\bar{n}} | H_1 | \psi^0_{\bar{n}} \rangle \cdot 1$$
$$\dot{C}_{n\neq\bar{n}}(t) = \frac{-i}{\hbar}\langle \psi^0_n | H_1 | \psi^0_{\bar{n}} \rangle e^{-i(E^0_{\bar{n}} - E^0_n)t/\hbar}$$
  + The solution is (work out normalization)
$$C_{\bar{n}}(t) = 1 - \frac{i}{\hbar}\int_0^t dt \, \langle \psi^0_{\bar{n}} | H_1(t) | \psi^0_{\bar{n}} \rangle \qquad \text{for } t \in T$$
$$C_{n\neq\bar{n}}(t) = \frac{-i}{\hbar}\int_0^t dt' \, \langle \psi^0_n | H_1(t') | \psi^0_{\bar{n}} \rangle e^{-i(E^0_{\bar{n}} - E^0_n)t'/\hbar}$$

– Periodic (aka Sinusoidal) Perturbations
  - As a very important example, take $H_1(t) = V e^{-i\omega t} + V^\dagger e^{i\omega t}$, $V =$ const. op.
    This could take the form $H_1 \propto \sin\omega t$ or $\cos\omega t$ or
    $H_1 \doteq V \begin{bmatrix} 0 & e^{\pm i\omega t} \\ e^{\mp i\omega t} & 0 \end{bmatrix}$, etc. However, at linear order, we can take only one complex exponential at a time in the amplitude.
  - Also, consider only 2 states $n = 1, 2$ with $\bar{n} = 1$.
  + Assume $\langle 1|V|1\rangle = \langle 2|V|2\rangle = 0$, $\langle 2|V|1\rangle = \langle 1|V^\dagger|2\rangle^* = V_{21}$
  + Define $\hbar\omega_0 = E^0_2 - E^0_1$. Defines a "natural" frequency
  - Solution:
$$C_1(t) = 1, \quad C_2(t) = \frac{V_{21}}{\hbar}\frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega}$$
  + Can become large when $\omega \simeq \omega_0$. (when is this valid?)
  + The $e^{+i\omega t}$ term does the same with $\omega_0 - \omega \to \omega_0 + \omega$, $V_{21} \to V^*_{12}$
    This is large when $\omega \simeq -\omega_0$, so we ignore it near $\omega \simeq \omega_0$
  + You also get a factor of $\frac{1}{2}$ if you use the sine or cosine form (or else redefine V_{12} as in the text)

- What does this mean?
  - + Transition Probability: the probability of measuring state 2 is $P = |\langle 2|\Psi(t)\rangle|^2 = |c_2(t)|^2$ by our usual rules
  - + For our sinusoidal perturbation,

$$P = \frac{4|V_{21}|^2}{\hbar^2}\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$



1) Oscillates in time.

2) Peaks at $\omega = \omega_0$, peak is higher at longer t

- + Key Point: At long times, transitions only happen for

$$E_2^0 - E_1^0 = \hbar\omega_0 = \pm\hbar\omega \leftarrow \text{perturbation carries energy for transition}$$

Will return on HW.

— Application to EM radiation. (Griffiths' discussion only heuristic)

- Recall that interaction with EM field is by Hamiltonian

$$H = \frac{(\vec{p}-q\vec{A})^2}{2m} + q\Phi$$

  - + Imagine $H_0 = p^2/2m + q\Phi_0$, Φ_0 = electrostatic potential
  - + Then $H_1 = \frac{-q}{2m}(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + \frac{q^2}{2m}\vec{A}^2 + q\Phi_1$, A, Φ_1 = 1st order

- An EM wave can be described by

$$\Phi_1 = 0, \quad \vec{\nabla}\cdot\vec{A} = 0, \ (\Rightarrow \vec{p}\cdot\vec{A} = \vec{A}\cdot\vec{p}), \quad \vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} + cc$$

  - + If $|q\vec{A}_0| \le$ 0th order energies, $H_1 \simeq \frac{q}{m}\vec{A}_0\cdot\vec{p}\, e^{-i\omega t}$

    we assume long wavelength so $\vec{k}\cdot\vec{x} \simeq 0$

- Now we can figure out transition probabilities using results from above.
- Many applications: photoelectric effect, lasers, etc. Also relates to full quantum theory of E+M, ...