PHYS-4601 Homework 6 Due 25 Oct 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Gaussian Wavepacket Part II** *based on Griffiths 2.22*

   Here we return to the Gaussian wavepacket in 1D, here looking at the time evolution for the free particle Hamiltonian. We recall from a previous assignment that the wavefunction (at some initial time) can be written as

   $$|\Psi(t=0)\rangle = \int_{-\infty}^{\infty} dx\ \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}|x\rangle = \int_{-\infty}^{\infty} dp\ \left(\frac{1}{2\pi a\hbar^2}\right)^{1/4} e^{-p^2/4a\hbar^2}|p\rangle\ . \tag{1}$$

   (a) Evolve this state in time. First write $\langle p|\Psi(t)\rangle$ and then show that

   $$\langle x|\Psi(t)\rangle = \frac{(2a/\pi)^{1/4}}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)}\ . \tag{2}$$

   Hint: You may want to use the trick of "completing the squares" to evaluate a Gaussian integral somewhere.

   (b) Find the probability density $|\langle x|\Psi(t)\rangle|^2$. Using the result from assignment 3 that $\langle x^2\rangle = 1/4a$ at $t=0$, find $\langle x^2\rangle$ at a later time t by inspection of the probability density. Qualitatively explain what's happening to the wavefunction as time passes.

   (c) What's the momentum-space probability density $|\langle p|\Psi(t)\rangle|^2$? Does $\langle p^2\rangle$ change in time? Does this state continue to saturate the Heisenberg uncertainty relation for $t \neq 0$?

2. **Proofs About Stationary States**

   (a) *Rephrasing Griffiths 2.1(c)* Consider the spatial part of a stationary state $\psi(x)$ (that is, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$) and suppose that the potential is an even function of x (ie, $V(x) = V(-x)$). Show that $\psi(\vec{x})$ can be chosen to be either an even or odd function of x. *Hint:* argue that, for any $\psi(x)$ that solves the time-independent Schrödinger equation, so does $\psi(-x)$. Use that to show that the even and odd parts $\psi_\pm(x) = [\psi(x) \pm \psi(-x)]/2$ are also solutions with the same energy.

   (b) *Griffiths 2.2 rephrased* Suppose that the energy E of a stationary state in one dimension is less than the minimum value of the potential. Use the time-independent Schrödinger equation to show that the second derivative of the wavefunction always has the same sign as the wavefunction. Then use that fact to argue qualitatively that such a wavefunction cannot be normalized, proving by contradiction that E must be greater than the minimum value of the potential.

3. **Dirac & the Wall**

   A particle moves in 1D in a potential

   $$V(x) = \begin{cases} \infty & x<0 \\ -\alpha\delta(x-d) & x>0 \end{cases} \quad \text{with } \alpha > 0\ . \tag{3}$$

   Note that the potential for $x<0$ means that the wavefunction satisfies the boundary condition $\psi(x=0) = 0$.

(a) Assuming that a bound state exists, show that the bound state energy is determined by a transcendental equation

$$\left(\frac{\hbar^2}{m\alpha d}\right) z = 1 - e^{-2z} \ , \tag{4}$$

where $z = (d/\hbar)\sqrt{-2mE}$.

(b) Using (4), find the condition that a bound state exists. *Hint:* think about the plots of the left-hand and right-hand sides of (4) and their behaviors near $z = 0$ and for large z.

(c) Show that the bound state energy is approximately

$$E = -\frac{m\alpha^2}{2\hbar^2} + \frac{m\alpha^2}{\hbar^2} e^{-2m\alpha d/\hbar^2} \tag{5}$$

for large d (when the wall is far from the delta function) by solving (4) iteratively, assuming that z is large. In otherwords, find a solution z_0 for (4) when you can drop the exponential, then find δz when $z = z_0 + \delta z$ is the solution and $z_0 \gg \delta z$. Finally, treat δz as small when you substitute back into the energy.

(d) Use Maple software to explore the solutions of (4). Attach a copy of your Maple code and output for the following steps. You should look up the different Maple commands in the help or online (google "maple [command name]" to get the help page) to learn about the options we are using.

1. Use Maple's `solve` command to find a general solution to (4) as follows: `solve(a*z = 1-exp(-2*z),z,allsolutions=true)`. You will find an answer in terms of the `LambertW` function. From now on, either set the first argument of `LambertW` to 0 or leave it out entirely. Note that $a = \hbar^2/m\alpha d$ and that $0 < a < 2$.
2. Now make a list of numerical solutions of (4) for 100 values of a for $0 < a < 2$, using the following code: `zlist:=[seq([i/50,fsolve((i/50)*z = 1-exp(-2*z), z, 0..2)],i=1..100)]:` The colon at the end hides the list, since it is long otherwise.
3. Now plot the solutions for z in three ways on the same plot: the exact solution from above, the numerical list of values, and the approximate solution $z = (1 - e^{-2/a})/a$ from the previous part. Use different colors for the exact and approximate solutions and circles for the list of values. You can use this code, where you should replace $f(a)$ with your exact solution above.

```
with(plots):
P1:=plot(f(a),a=0..2,color=green)
P2:=listplot(zlist,style=point, symbol=circle)
P3:=plot((1-exp(-2/a))/a,a=0..2,color=red)
display([P1,P2,P3],labels=[a,z])
```

You should see that the green (exact) curve matches the numerical points very well, but the red approximate curve is different for larger values of a.