PHYS-4601 Homework 5 Due 18 Oct 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **The Virial Theorem** *Based on Griffiths 3.31*

   Consider 3D quantum mechanics.

   (a) Using Ehrenfest's theorem, show that

   $$\frac{d}{dt}\langle\vec{x}\cdot\vec{p}\rangle = \left\langle\frac{\vec{p}^2}{m}\right\rangle - \left\langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle \ . \tag{1}$$

   If the system is in a *stationary state*, recall from the lecture notes that all expectation values are time-independent, so the left-hand side of (1) vanishes. This gives the *virial theorm*

   $$2\langle K\rangle = \left\langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle \ , \tag{2}$$

   where K is the kinetic energy. (The virial theorem holds classically, also, though without the expectation values.)

   (b) Using the virial theorem, find the ratio of (the expectation value of) the kinetic energy to the potential energy for the harmonic oscillator potential $V\propto\vec{x}^2$ and for the Coulomb potential $V\propto 1/|\vec{x}|$. *Hint:* Remember from electromagnetism (or Newton's law of gravity) that $\vec{\nabla}(1/|\vec{x}|) = -\vec{x}/|\vec{x}|^3$.

2. **Matrix Hamiltonian**

   Consider a 3D Hilbert space with Hamiltonian

   $$H \simeq E_0\begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{3}$$

   in some basis. Work in this basis throughout the problem.

   (a) Show that the time evolution operator is

   $$e^{-iHt/\hbar} \simeq \cos(E_0t/\hbar)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - i\sin(E_0t/\hbar)\begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{4}$$

   in this basis.

   (b) Some operator A is defined in this basis as

   $$A \simeq A_0\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ . \tag{5}$$

   Suppose the system starts out at time $t=0$ in a state represented by $[1\ 0\ 0]^T$. Using your previous result, find the state of the system and $\langle A\rangle$ as a function of time. At what times is $\langle A\rangle$ minimized?

3. **Kaon Oscillation**

Kaons are subatomic particles; there are two neutral kaons. Like neutrinos, they can oscillate. In the kaon mass eigenbasis $\{|e_1\rangle, |e_2\rangle\}$, the Hamiltonian can be written as the matrix

$$H \simeq \begin{bmatrix} (m+\Delta m)c^2 & 0 \\ 0 & (m-\Delta m)c^2 \end{bmatrix} . \tag{6}$$

The flavor eigenstates are

$$|f_1\rangle = \cos\theta|e_1\rangle + \sin\theta|e_2\rangle \ , \quad |f_2\rangle = -\sin\theta|e_1\rangle + \cos\theta|e_2\rangle \tag{7}$$

for some constant θ.

(a) If a kaon is initially in flavor state $|f_1\rangle$, the probability of measuring it in flavor state $|f_2\rangle$ is a periodic function of time. Find the frequency of that oscillation probability.

(b) What is the uncertainty of the energy for the state $|f_1\rangle$?

(c) Let P_{max} be the maximum oscillation probability. Then we expect one oscillation for every $1/P_{max}$ periods of $P(t)$. Use this length of time as Δt and argue that kaon oscillations respect the energy-time uncertainty principle.