# PHYS-4601 Homework 4 Due 4 Oct 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Commutators and Functions of Operators**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of some operator $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. Consider the inverse operator $\mathcal{O}^{-1}$ defined such that $\mathcal{O}\mathcal{O}^{-1} = \mathcal{O}^{-1}\mathcal{O} = 1$. Show that $|\lambda\rangle$ is an eigenvector of $\mathcal{O}^{-1}$ with eigenvalue $1/\lambda$.

   (b) For any function $f(x)$ that can be written as a power series

   $$f(x) = \sum_n f_n x^n \ , \tag{1}$$

   we can define

   $$f(\mathcal{O}) = \sum_n f_n \mathcal{O}^n \ , \tag{2}$$

   where $\mathcal{O}^n$ denotes operating with $\mathcal{O}$ $n$ times. Show that

   $$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \ . \tag{3}$$

   Does this result hold if the power series includes negative powers?

   (c) For any three operators $A, B, C$, show that

   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{4}$$

   (d) Then prove by induction that

   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{5}$$

   if $[A, B]$ commutes with $B$ (for $n > 0$).

   (e) Finally, show using (5) that $[p, f(x)] = -i\hbar df/dx$, where $x$ and $p$ are 1D position and momentum operators. Assume $f(x)$ can be written as a Taylor series.

2. **Measurement vs Time Evolution** *a considerable expansion of Griffiths 3.27*

   Suppose a system has observable $A$ with eigenstates $|a_1\rangle, |a_2\rangle$ of eigenvalues $a_1, a_2$ respectively and Hamiltonian $H$ with eigenstates $|E_1\rangle, |E_2\rangle$ of energies $E_1, E_2$ respectively. The eigenstates are related by

   $$|a_1\rangle = \frac{1}{5}\left(3|E_1\rangle + 4|E_2\rangle\right) \ , \quad |a_2\rangle = \frac{1}{5}\left(4|E_1\rangle - 3|E_2\rangle\right) \ . \tag{6}$$

   Suppose the system is measured to have value $a_1$ for $A$ initially. Each of the following parts asks about a different possible set of subsequent measurements.

   (a) What is the probability of measuring energy $E_1$ immediately after the first measurement? Assuming we do get $E_1$, what is the probability of measuring $a_1$ again if we measure $A$ again immediately after the measurement of energy?

   (b) Instead, consider immediately measuring $A$ again after the first measurement. What are the probabilities for observing $a_1$ and $a_2$?

(c) Finally, consider making the first measurement and then allowing the system to evolve for time $t$. If we then measure energy, what is the probability of finding energy $E_1$? If we instead measured $A$ again, what is the probability we find $a_1$ again?

3. **Time Dependence & Schrödinger in 1D**

   Consider motion in 1 dimension with the usual Hamiltonian $H = p^2/2m + V(x)$. In each part, you are given a normalized time-dependent wavefunction. Answer the following questions.

   (a) *inspired by Reed and an MIT question* The particle has wavefunction

$$\Psi(x,t) = \begin{cases} 0 & (x < 0) \\ 2k^{3/2} x e^{-kx} e^{i\hbar k^2 t/2m} & (x > 0) \end{cases}, \tag{7}$$

   where $k$ is a constant. What are the potential $V(x)$ and energy $E$ of the particle?

   (b) The particle has wavefunction

$$\Psi(x,t) = \begin{cases} 0 & (x < 0) \\ Ae^{-i\alpha t} \sin\left(\frac{\pi x}{L}\right)\left[1 + e^{-i\beta t}\cos\left(\frac{\pi x}{L}\right)\right] & (0 < x < L) \\ 0 & (x > L) \end{cases}, \tag{8}$$

   where $A, \alpha, \beta$ are constants, and the potential is constant for $0 < x < L$. What is $\beta$? Can you determine $\alpha$?