PHYS-4601 Homework 3 Due 27 Sept 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dyad to Matrix**

   Let $\{|1\rangle, |2\rangle, |3\rangle\}$ be an orthonormal basis and the operator

   $$Q = |1\rangle\langle 1| - i|2\rangle\langle 1| + i|1\rangle\langle 2| + 2i|3\rangle\langle 2| - 2i|2\rangle\langle 3| + 4|3\rangle\langle 3| \ . \tag{1}$$

   (a) Find the matrix element $\langle 3|Q^3|1\rangle$.

   (b) Write Q as a matrix in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis. This is the basis where

   $$|1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \quad |2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \quad |3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ . \tag{2}$$

   (c) Finally, define a new basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ with

   $$|e_1\rangle = \frac{|1\rangle + 2|3\rangle}{\sqrt{5}} \ , \quad |e_2\rangle = |2\rangle \ , \quad |e_3\rangle = \frac{2|1\rangle - |3\rangle}{\sqrt{5}} \tag{3}$$

   (I have checked for you that this basis is orthonormal). First write Q as a dyad operator in terms of this new basis and then write Q as a matrix in this basis.

2. **Expectation and Uncertainty**

   Consider an observable L with three eigenvalues +1, 0, and -1 and corresponding eigenstates $|+1\rangle, |0\rangle, |-1\rangle$. We have a system in state

   $$|\psi\rangle = \frac{1}{3}\Big(|+1\rangle + 2e^{i\beta}|0\rangle + 2|-1\rangle\Big) \ . \tag{4}$$

   (a) What is the probability of measuring each of the three eigenvalues of L?

   (b) Find the expectation value and uncertainty of a measurement of L.

   (c) Another observable A acts on the L eigenbasis as

   $$A|+1\rangle = \frac{1}{\sqrt{2}}|0\rangle \ , \quad A|0\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle + |-1\rangle\right) \ , \quad A|-1\rangle = \frac{1}{\sqrt{2}}|0\rangle \ . \tag{5}$$

   Find the expectation value and uncertainty of A in state $|\psi\rangle$.

3. **Gaussian Wavepacket Part I**

   Here we take a first look at the Gaussian wavepacket in 1D, which is an important state in more than one physical system. In this problem, we will consider the state at a single instant $t = 0$, ignoring its time evolution. The state is

   $$|\psi\rangle = \int_{-\infty}^{\infty} dx \ Ae^{-ax^2}|x\rangle \ . \tag{6}$$

   Note that these results will be useful in future assignments.

(a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates. The textbook cover also has a formula for Gaussian integrals.

(b) Since the wavefunction is even, $\langle x \rangle = 0$. Find $\langle x^2 \rangle$. *Hint:* You can get a factor of x^2 next to the Gaussian by differentiating it with respect to the parameter a.

(c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

(d) Find $\langle p \rangle$ and $\langle p^2 \rangle$ and show that this state saturates the Heisenberg uncertainty principle.