PHYS-4601 Homework 20 Due 28 Mar 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Fermi's Golden Rule**

   Consider a sinusoidal perturbation Hamiltonian $H_1 = Ve^{-i\omega t} + V^\dagger e^{+i\omega t}$. In the class notes, we found the probability for a transition from state $|1\rangle$ to $|2\rangle$ as a function of time and frequency ω. In the following, define $\hbar\omega_0 = E_2 - E_1$, the difference of the energy eigenvalues of the unperturbed Hamiltonian H_0. We will investigate the transition probability near $\omega = \omega_0$ at large t (at least as long as the probability stays small).

   (a) At a fixed (and large) time, the probability is peaked at $\omega = \omega_0$. Using L'Hospital's rule or just a power series expansion, find the peak transition probability as a function of time.

   (b) Find the values of ω where the probability first vanishes on either side of $\omega = \omega_0$. The difference in these two values tells us the width of the peak.

   (c) For large enough times, approximate the transition probability as a rectangle with the peak value from part (a) and width given by half the difference in part (b). Integrate this approximate probability function and argue that

   $$P \to \frac{2\pi|V_{21}|^2}{\hbar^2} t\delta(\omega_0 - \omega) \tag{1}$$

   as $t \to \infty$.

   This problem shows two things: first, transitions occur only to states at energies related by the perturbation frequency and, second, that there is a constant transition rate (probability per unit time) to the appropriate states. The relationship (1) is known as *Fermi's Golden Rule*. (There is of course a more rigorous derivation possible.)

2. **Exciting a 3D Harmonic Oscillator**

   Consider an electron moving in a 3D harmonic oscillator potential with Hamiltonian

   $$H_0 = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \ . \tag{2}$$

   Starting at time $t = 0$, the electron is exposed to a weak electromagnetic wave moving along z and polarized along x, which introduces a term $H_1(t) = (E_0/\omega)p_x \exp(ikz - i\omega t)$ (plus complex conjugate) to the Hamiltonian. The wavelength is long, so you can approximate that $kz \ll 1$. Recall that the eigenstates of H_0 can be written in terms of x, y, z excitation numbers as $|n_x, n_y, n_z\rangle$ with energies $\hbar\omega_0(n_x + n_y + n_z - 3/2)$.

   (a) Let P_n be the probability that an electron initially in harmonic oscillator state $|n, 0, 0\rangle$ transitions to state $|n + 1, 0, 0\rangle$ at time T. Find the ratio P_n/P_0. You may approximate that the EM wave is spatially uniform.

   (b) In the approximation that the EM wave is spatially uniform, the excitation from $|0, 0, 0\rangle$ to $|1, 0, 1\rangle$ is forbidden (the transition probability vanishes). Using instead the approximation that $\exp(ikz) \sim (1 + ikz)$, find the probability of that transition at time T.

3. **Variational Principle for the Linear Well**

   Consider a particle moving in 1D in a potential $V(x) = \alpha|x|$. Find the best possible upper bound on the ground state energy using a gaussian trial wavefunction.

4. **Perturbation Theory vs Variational Principle**

   (a) Consider a particle moving in the 1D anharmonic oscillator potential

   $$V(x) = \frac{1}{2}m\omega^2 x^2 + gx^3 \tag{3}$$

   First, consider perturbations of the harmonic oscillator ground state $|0\rangle$ and find the ground state energy to first order in g (*Hint:* this should be a simple calculation). Then, using the variational method, show that the true ground state energy of this potential is unbounded below (that is, if I give you any real number, demonstrate that the ground state energy is less than that number). We say that this potential is unstable and has no ground state. *Hint*: Think about a simple trial wavefunction that approximates a delta function in position.

   (b) *from Griffiths 7.5* Consider a Hamiltonian $H = H_0 + H_1$, where H_0 is exactly solvable and H_1 is small in some sense. Prove that first-order perturbation theory always overestimates the true ground state energy. That is, show that the ground state energy calculated in first-order perturbation theory is greater than (or equal to) the true ground state energy.