PHYS-4601 Homework 2 Due 20 Sept 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Diagonalization** *Based on Griffiths A.26*

   Consider a three-dimensional Hilbert space with orthonormal basis $|e_i\rangle$, $i = 1, 2, 3$. The operator A takes the matrix representation

   $$A = \sum_{i,j} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j| \simeq \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \tag{1}$$

   You should be able to check yourself that A is Hermitian.

   (a) Find the eigenvalues a_i and corresponding eigenstates $|a_i\rangle$ ($A|a_i\rangle = a_i|a_i\rangle$) written in terms of their components $\langle e_j|a_i\rangle$. Choose the eigenstates to form an orthonormal eigenbasis; that is, choose any ambiguities such that $\langle a_i|a_j\rangle = \delta_{ij}$.

   (b) As we will state in class, A can be written in the form

   $$A = \sum_i a_i |a_i\rangle\langle a_i| \ , \tag{2}$$

   where a_i are the eigenvalues and $|a_i\rangle$ are the eigenvectors of A. Verify that formula (2) gives the same operator as (1) when you plug in your answer to part (a) for the eigenvalues and eigenvectors.

2. **The Momentum Operator**

   The usual definition of the momentum operator p is that it acts by $p \simeq -i\hbar d/dx$ in the position basis (precisely, $\langle x|p|\psi\rangle = -i\hbar d\psi/dx$ for any state $|\psi\rangle$ with wavefunction $\langle x|\psi\rangle = \psi(x)$). In this problem, we will explore some properties of this operator and its eigenfunctions; in the future, we will see why it makes sense to call it momentum. For simplicity, we work in one dimension with $-\infty < x < \infty$.

   (a) Let the state $|p\rangle$ be an eigenstate of the momentum operator with real eigenvalue p (ie, $p \cdot |p\rangle = p|p\rangle$). Show that $|p\rangle$ has wavefunction

   $$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{3}$$

   (you may assume the normalization constant is given).

   (b) Show that $\langle p'|p\rangle = \delta(p - p')$. *Hint:* You may find the formula

   $$\delta(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikz} \tag{4}$$

   helpful.

   (c) Show that the wavefunction $\psi(x) = \langle x|\psi\rangle$ and "momentum-space wavefunction" $\tilde{\psi}(p) = \langle p|\psi\rangle$ for any vector $|\psi\rangle$ are Fourier transforms, as defined in Griffiths equation [2.102] (up to factors of $\hbar$). To work this out precisely, it will be helpful for you to rescale x and p to remove explicit powers of $\hbar$.

What we have seen so far is that $p \simeq -i\hbar d/dx$ has complex exponentials for eigenfunctions, and that this implies that $\langle x|\psi\rangle$ and $\langle p|\psi\rangle$ are Fourier transforms for any state $|\psi\rangle$. Now we want to prove the reverse: assuming (3), we will show that $p \simeq -i\hbar d/dx$ in the position eigenbasis.

(d) For momentum to be observable, it must be Hermitian. Assuming p is a Hermitian operator, show $\langle p|p|\psi\rangle = p\tilde{\psi}(p)$ for any state $|\psi\rangle$. In other words, you are showing that $p \cdot \tilde{\psi}(p) = p\tilde{\psi}(p)$ on momentum-space wavefuntions (like $x \cdot \psi(x) = x\psi(x)$ on normal wavefunctions).

(e) Assuming (3), we know that the wavefunction and momentum-space wavefunction are Fourier transforms of each other. Use this fact to show that

$$\langle x|p|\psi\rangle = -i\hbar \frac{d\psi}{dx}(x) \ . \tag{5}$$

What this means is that defining $p \simeq -i\hbar d/dx$ is equivalent to defining the state $|p\rangle$ by (3) — you can derive one statement from the other.

3. **The Last Eigenvector**

A system with a three-dimensional Hilbert space has Hamiltonian represented by the matrix

$$H \simeq \frac{E_0}{3} \begin{bmatrix} 1/2 & -1/2 & 2i \\ -1/2 & 1/2 & -2i \\ -2i & 2i & -1 \end{bmatrix} \ . \tag{6}$$

Two of the eigenstates are represented by the column vectors

$$|1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } |2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} \ . \tag{7}$$

(a) Find the third eigenstate $|3\rangle$ as a column vector, including proper normalization.

(b) Find all the eigenvalues of H [EXTRA CREDIT: and write H as a matrix in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.]

(c) Write the projection operator $|3\rangle\langle 3|$ as a matrix in the original basis [EXTRA CREDIT: and the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.]