PHYS-4601 Homework 19 Due 21 Mar 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Estimating Helium Better** *Griffiths 5.11 clarified*

   In this problem, we will estimate the ground state energy of a helium atom. We treat the electron repulsion as a first-order correction to the attraction between the electrons and the nucleus.

   (a) Consider the states of a single electron around a helium nucleus (which has twice the charge of a proton). Argue that the "helium Bohr radius" $a_{\text{He}} = a/2$, where a is the usual Bohr radius, and that therefore the single-electron ground state wavefunction is given by

   $$\langle \vec{x} | n = 1, \ell = 0, m = 0 \rangle = \sqrt{\frac{8}{\pi a^3}} e^{-2r/a} \ . \tag{1}$$

   Next assume that the two electron helium groundstate is $|n = 1, \ell = 0, m = 0\rangle_1 |n = 1, \ell = 0, m = 0\rangle_2 |s = 0, m_s = 0\rangle$, where the total spin state is the antisymmetric singlet. (The spatial wavefunction is given by Griffiths eqn [5.30].) Briefly argue that the energy of this state, in the absence of electron repulsion, is given by Griffiths eqn [5.31].

   (b) Now find $\langle |\vec{x}_1 - \vec{x}_2|^{-1} \rangle$ in this state, as follows:

   1. Use the trick of setting the z axis for $\vec{x}_2$ along $\vec{x}_1$ and the law of cosines to see $|\vec{x}_1 - \vec{x}_2| = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2}$.
   2. Do the angular integrals for $\vec{x}_2$, noting that

      $$\int_0^\pi d\theta \sin\theta f(\cos\theta) = \int_{-1}^1 dx f(x) \ .$$

      Your result will have square roots of perfect squares, which are equal to absolute values. *Be careful of that!*
   3. Carry out the r_2 integral in two parts, $0 < r_2 \leq r_1$ and $r_1 < r_2 < \infty$.
   4. Now do the $\vec{x}_1$ integrals.

   Hint: The "exponential integrals" formula in the back cover of Griffiths will be helpful.

   (c) Use your result to find the change in ground state energy ΔE at first order in perturbation theory. Write ΔE in terms of the Bohr radius a and estimate its value in eV. Then add this to the energy from part (a) to get a rough estimate of the He ground state energy. *Hint*: Remember that the hydrogen ground state energy is $-\hbar^2/2ma^2 = -13.6$ eV.

2. **Matrix Perturbation Theory**

   Consider the matrix Hamiltonian

   $$H \simeq \begin{bmatrix} E_1 & \epsilon \\ \epsilon & E_2 \end{bmatrix} \tag{2}$$

   with $E_1 \neq E_2$ except when you are told otherwise. Assume that $\epsilon \ll E_1, E_2$.

   (a) To first order in perturbation theory, find the energy eigenvalues and eigenstates.

   (b) What is the first order correction to the energy if $E_1 = E_2 = E$?

   (c) Find the energy eigenvalues to second order in perturbation theory.

(d) Find the energy eigenvalues and eigenstates exactly. Then expand them as a power series in ϵ and compare to your perturbative answers from parts (a,c). In the case that $E_1 = E_2 = E$, how does your answer compare to part (b)?

3. **Weak-Field Zeeman Effect**

In the class notes, we stated that placing a hydrogen atom in a constant magnetic field $B_0\hat{z}$ introduces a contribution to the hydrogen atom of $H_1 = (e/2m)B_0(L_z + 2S_z)$. If this contribution is larger than the energy level splitting due to fine structure, this gives the "strong-field" Zeeman effect that we discussed in class. In this problem, consider the opposite limit, in which H_1 is smaller than the fine structure splitting. In this case, we include the fine structure corrections in the "unperturbed" Hamiltonian H_0 and treat H_1 as the perturbation to that.

(a) With fine structure included, the eigenstates of H_0 are identified by n, total angular momentum quantum number j, its z component m_j, and the total orbital angular momentum quantum number ℓ (as well as total spin $s = 1/2$); the z-components m_ℓ and m_s are not good quantum numbers. Write $H_1 = (e/2m)B_0(J_z + S_z)$ since $\vec{J} = \vec{L} + \vec{S}$ and show that the change in energy due to B_0 is

$$E^1_{n,j,m_j,\ell} = \frac{e\hbar}{2m} B_0 m_j \left[1 \pm \frac{1}{2\ell+1}\right] . \tag{3}$$

To do this, you will need to know that the eigenstate of J^2, J_z, and L^2 is written

$$\begin{aligned} |j = \ell \pm 1/2, m_j, \ell\rangle &= \sqrt{\frac{\ell \mp m_j + 1/2}{2\ell+1}} |\ell, m_\ell = m_j + 1/2, m_s = -1/2\rangle \\ &\pm \sqrt{\frac{\ell \pm m_j + 1/2}{2\ell+1}} |\ell, m_\ell = m_j - 1/2, m_s = 1/2\rangle \end{aligned} \tag{4}$$

in terms of the eigenstates of L^2, L_z, and S_z. *Hint:* It may be useful to note that $[H_0, J_z] = [H_1, J_z] = 0$.

(b) The quantity in square brackets in (3) is called the Landé g factor. Show that the g factor can also be written as

$$\left[1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}\right] , \tag{5}$$

which is the form given in Griffiths. You can start with (5) and try $j = \ell \pm 1/2$ separately to get the form given in (3).