PHYS-4601 Homework 18 Due 14 Mar 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Not-Quite-Square Well**

   Consider a particle moving in a 1D well of potential

$$\begin{cases} V_0 x/a & 0 < x < a \\ \infty & \text{otherwise} \end{cases} . \tag{1}$$

   Assume that $\epsilon = ma^2 V_0/\hbar^2 \ll 1$. Recall that the energy eigenfunctions and eigenvalues are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \ , \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 \ , \quad n = 1, 2, \cdots \ . \tag{2}$$

   (a) Show that the first order contribution to the energy is $E_n^1 = V_0/2$ for all n.

   (b) Now consider the ground state of the system. Recalling that the first order correction to the ground state can be written as

$$|\psi_1^1\rangle = \sum_{n=2}^{\infty} c_n |\psi_n^0\rangle \ , \tag{3}$$

   use Maple's `seq` and `int` commands to make a list of the coefficients c_n for $n = 2, \cdots 10$. Attach a copy of your Maple code. You should work in units where $a = 1$ and express your answer in terms of the parameter ϵ.

   (c) Use Maple to plot the uncorrected ground state wavefunction and the wavefunction with first order terms (including corrections from the $n = 2, \cdots 10$ states) on the same plot. In order to see the difference, use an exaggerated value of $\epsilon = 3$.

2. **Relativistic Harmonic Oscillator**

   Recall that the relativistic energy is $\sqrt{(\vec{p}c)^2 + (mc^2)^2} \approx mc^2 + \vec{p}^2/2m - \vec{p}^4/8m^3c^2$ (plus higher-order corrections), so a 1D harmonic oscillator with the first relativistic correction has the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \frac{1}{2}m\omega^2 x^2 \ . \tag{4}$$

   (a) Find the ground state energy of this oscillator using first-order perturbation theory. What condition must the frequency satisfy for the relativistic correction to be small?

   (b) Find the ground state eigenstate of this oscillator in terms of the unperturbed oscillator eigenstates using first-order perturbation theory.

3. **Stark Effect** *based on Griffiths 6.36*

   The presence of an external electric field $E_0\hat{z}$ shifts the energy levels of a hydrogen atom, which is called the Stark effect. Consider the hydrogen atom to be described by the Coulomb potential; the external electric field introduces a perturbation

$$H_1 = eE_0 z = eE_0 r\cos\theta \ . \tag{5}$$

We have already seen on homework that the expectation value of this Hamiltonian in the ground state $n = 1$ vanishes, so there is no shift in the ground state energy. In this problem, we consider the degenerate perturbation theory of the $n = 2$ states. *As spin does not enter, do not consider it in this problem.*

(a) The four states $|2,0,0\rangle$, $|2,1,0\rangle$, and $|2,1,\pm1\rangle$ are degenerate at 0th order. Label these states sequentially as $i = 1,2,3,4$. Show that the matrix elements $W_{ij} = \langle i|H_1|j\rangle$ form the matrix

$$W = -3aeE_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

where empty elements are zero and a is the Bohr radius. *Hint*: Note that L_z commutes with H_1, so only states with the same quantum number m can have nonzero matrix elements; this will save you quite a bit of work. Then use the angular wavefunctions to see that all the diagonal elements of W must vanish. Finally, use the explicit wavefunctions to evaluate the remaining matrix elements of W (there should only be one independent one left).

(b) Diagonalize this matrix to show that $|\pm\rangle = (1/\sqrt{2})(|2,0,0\rangle \pm |2,1,0\rangle)$ are eigenstates of W. Find the first order shift in energies of $|\pm\rangle$. *Hint*: Note that the corrected eigenstates may still have contributions from other values of the principal quantum numbers n, but that doesn't quite matter.

(c) Finally, show that the states $|\pm\rangle$ have a nonzero dipole moment $p_z = -e\langle z\rangle$ and calculate it. You should not need to do any more calculations; just use your answer from part (b).