# PHYS-4601 Homework 15 Due 16 Feb 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

*You may set Boltzmann's constant $k_B = 1$ for this assignment if you wish.*

1. **White Dwarfs** *based on Griffiths 5.35*

   White dwarfs are old, dead stars that don't support nuclear fusion any more. Instead, electron degeneracy pressure keeps them from collapsing into black holes. In this problem, you'll work out the white dwarf radius by finding the minimum energy. *Note:* You will need equation [5.45] from Griffiths.

   (a) Imagine that the white dwarf is a sphere of uniform density. Caculate the gravitational potential energy $V$ in terms of its radius $R$, total mass $M$, and Newton's constant $G$. *Hint*: To do this, work out the potential energy of a shell of thickness $dr$ at radius $r$, which comes from the sphere of matter inside that shell. Then integrate from $r = 0 \to R$. You should find $V = -3GM^2/5R$.

   (b) Now treat the electrons as a free electron gas. Write their total energy (which is kinetic, found in the reading) in terms of $M$, $R$, the electron mass $m$, the nucleon mass $m_N$, the number of electrons per nucleon $q$, and $\hbar$. Then add this to the result of part (a) to get the total energy and find the radius $R$ where the energy is minimized.

   (c) Assuming $q \approx 1/2$ (the average nucleus is a helium nucleus), find the radius (in km) of a white dwarf with the mass of the sun. You will need $G = 7\times10^{-11}$ m$^3$/kg/s$^2$, $\hbar = 1\times10^{-34}$ Js, $m = 9 \times 10^{-31}$ kg, $m_N = 2 \times 10^{-27}$ kg, and $M = 2 \times 10^{30}$ kg.

   (d) A neutron star is an extremely dense star where all the matter is neutrons, which is supported against collaps by the degeneracy pressure of the neutrons. In this case, $q = 1$. Find the ratio of the radius of a neutron star to the radius of a white dwarf of the same mass (assume $q \approx 1$ for the white dwarf), both in terms of physical constants and as a pure number to 1 significant digit.

2. **Band Structure** *from Griffiths & Schroeter, 3rd ed, 5.28*

   In class, we showed that the energy eigenvalues for the Dirac comb potential with periodic boundary conditions are the solutions of

   $$\cos(qa) = \cos(ka) + \frac{m\alpha}{\hbar^2 k}\sin(ka)\ , \quad k = \sqrt{2mE}/\hbar\ , \tag{1}$$

   where $qa$ is a large number of discrete values between 0 and $2\pi$.

   (a) Define $x = qa$ to be $q$ in units of $1/a$ and $y = E/(\hbar^2/2ma^2)$ the energy in units of $\hbar^2/2ma^2$. Write equation (1) in terms of $x$, $y$, and the dimensionless combination $z = \hbar^2/2ma\alpha$.

   (b) The Maple `implicitplot` command shows all the values of a variable$y$ that solve a specified relationship for each value of variable $x$. Use `implicitplot` to plot the allowed energies $y$ as a function of wavenumber $x$ in the range $0 \le x \le 2\pi$ and $0 < y < 100$ for values $z = 1/100, 1/20, 1/2, 1, 5$ (a separate plot for each value of $z$). Attach a printout of your Maple code with the plots. *Hint:* You can look up the syntax for `implicitplot` in the Maple help. You will need to run `with(plots):` in order to load the `implicitplot` command.

(c) Based on your plots, for what values of $q$ are the first and second bands closest to each other? The second and third bands? (This tells us where the gaps are smallest.) What happens to the bands as the potential strength $\alpha$ decreases?

3. **Bose-Einstein Condensation** *from Griffiths 5.29*

Consider the Bose-Einstein distribution for spin-0 bosons.

(a) From our class discussion on the free electron gas, argue that the number of states of a free spin-0 particle with wavevector magnitude between $k$ and $k + dk$ in a volume $V$ is

$$\Omega(k)dk = \frac{V}{2\pi^2}k^2 dk \ , \tag{2}$$

which is known as the density of states. (You can just quote results from the class notes.)

(b) If the particles occupy states with large wavenumbers, argue that the number density and energy density are given by

$$n = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 f_{BE}(x) \ , \quad \rho = \frac{\hbar^2}{4m\pi^2}\int_0^\infty dk\, k^4 f_{BE}(x) \ , \quad x = \frac{1}{T}\left(\frac{\hbar^2 k^2}{2m} - \mu\right) \ , \tag{3}$$

where $f_{BE}$ is the Bose-Einstein distribution $f_{BE}(x) = 1/(e^x - 1)$. Here $m$ is the particle mass and $\mu$ is the chemical potential.

(c) Consider the case that $\mu = 0$. If there is a fixed number density $n$ of bosons, this occurs at a fixed critical temperature $T_c$; $\mu < 0$ for higher temperatures, and the integral approximation above fails for lower temperatures because all the bosons go into the ground state (*Bose-Einstein condensation*). Find $T_c$ as a function of $n$. *Hint:* the integral

$$\int_0^\infty dx\, \frac{x^{s-1}}{e^x - 1} = \Gamma(s)\zeta(s) \ , \tag{4}$$

in terms of the gamma function and Riemann zeta function. You may write your answer in terms of these functions (that is, you don't need to give the numerical value for those functions).