PHYS-4601 Homework 12 Due 17 Jan 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Electromagnetic Gauge Transformations** *from Griffiths 4.61*

   Recall from class that the Hamiltonian for a particle of charge q is

   $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi \ , \tag{1}$$

   where the potential is Φ and vector potential is $\vec{A}$. The electric and magnetic field are

   $$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{2}$$

   (a) Show that the electromagnetic fields are invariant under *gauge transformations*. That is, show that the potentials

   $$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \ , \quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{3}$$

   give the same $\vec{E}$ and $\vec{B}$ fields as Φ and $\vec{A}$, where Λ is any function of $\vec{x}$ and t.

   (b) Assume that a wavefunction $\Psi(\vec{x}, t)$ solves the time-dependent Schrödinger equation for potentials Φ and $\vec{A}$. Show that

   $$\Psi' = e^{iq\Lambda/\hbar}\Psi \tag{4}$$

   solves the time-dependent Schrödinger equation for the potentials Φ' and $\vec{A}'$ given in (3). This shows that quantum physics also respects gauge transformations.

   This gauge invariance is a critical feature of the quantum mechanical theory of electromagnetism with profound consequences. We may explore aspects of it again in assignments.

2. **Landau Levels** *adapted from Griffiths 4.60*

   This problem considers the motion of electrons which are essentially confined to a 2D surface in the presence of an orthogonal magnetic field. This is the system used to describe the quantum Hall effect. Since this is a 2D problem, we won't include p_z (if you like, you can imagine that we consider only eigenstates of p_z with zero eigenvalue).

   (a) Show that a magnetic field $\vec{B} = B_0\hat{k}$ can be described by vector potential $\vec{A} = (B_0/2)(x\hat{\jmath} - y\hat{\imath})$. ($\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors along x, y, z respectively.)

   (b) Show that we can write

   $$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) - \omega L_z \ , \tag{5}$$

   where $\omega = qB_0/2m$. Argue that $[H, L_z] = 0$ (you may use results from earlier problems).

   (c) Except for the L_z term at the end, this looks like a harmonic oscillator in the x and y directions. Write H and L_z in terms of the raising and lowering operators $a_x^\dagger, a_y^\dagger, a_x, a_y$ of those two harmonic oscillators. Evaluate $L_z|n_x, n_y\rangle$; is it diagonal?

(d) Apparently we have not yet found how to diagonalize H and L_z simultaneously. Now define lower operators

$$A = \frac{1}{\sqrt{2}}(a_y + ia_x) \ , \quad \bar{A} = \frac{1}{\sqrt{2}}(a_y - ia_x) \tag{6}$$

and their adjoints, the raising operators. First, show that $A, A^\dagger$ and $\bar{A}, \bar{A}^\dagger$ satisfy the usual commutation relations for raising and lowering operators. Then, find H and L_z in terms of $A, \bar{A}, A^\dagger, \bar{A}^\dagger$. From those expressions, argue that the energy eigenvalues are $E_n = \hbar\omega_B(n + 1/2)$, where $\omega_B = 2\omega = qB_0/m$ is the *cyclotron frequency*, and that the energy eigenstates are infinitely degenerate. These energy levels are called *Landau levels* (the stationary states of electrons in a piece of metal in a magnetic field, relevant for the Hall effect); in practice, the finite size of the metal where the electrons live reduces the degeneracy to a finite amount.

3. **MRI Physics** *Inspired by a question by Horbatsch*

Consider a spin-1/2 proton with gyromagnetic ratio γ in the presence of a magnetic field

$$\vec{B} = B_0\hat{z} + B_1\cos(\omega t)\hat{x} - B_1\sin(\omega t)\hat{y} \tag{7}$$

at its fixed position. This is a magnetic field with a fixed z component and another component rotating in the x, y plane.

(a) Write the Hamiltonian in terms of spin operators and then as a matrix in the eigenbasis of the S_z operator.

(b) It is actually possible to find the full time-dependent state for this system. If the spin is up at $t = 0$, the solution of the time-dependent Schrödinger equation is

$$\begin{aligned} \langle +|\Psi(t)\rangle &= e^{i\omega t/2}\left[\cos(\alpha t/2) - i\frac{(\omega - \gamma B_0)}{\alpha}\sin(\alpha t/2)\right] \\ \langle -|\Psi(t)\rangle &= ie^{-i\omega t/2}\frac{\gamma B_1}{\alpha}\sin(\alpha t/2) \end{aligned} \tag{8}$$

with $\alpha = \sqrt{\gamma^2 B_1^2 + (\omega - \gamma B_0)^2}$. Use Maple to verify that (8) solves the Schrödinger equation. Input the Schrödinger equation and initial conditions as a list of equations and then the solution above as another list. Then use the `odetest` function in Maple to check that (8) solves the time-dependent Schrödinger equation. Include a copy of your Maple code. You may want to use the Maple help to learn how to use `odetest`.

(c) Use (8) to find the transition probability from spin up ($|+\rangle$) to spin down ($|-\rangle$) as a function of time. Find the conditions that this probability is one.