PHYS-4601 Homework 11 Due 10 Jan 2019

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Raising and Lowering**

   (a) *More or less Griffiths 4.18* Using the relation for $L_\pm L_\mp$ given in class and the text, show that
   $$L_\pm|\ell,m\rangle = \hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}|\ell,m\pm 1\rangle\ . \tag{1}$$

   (b) In a vector/matrix representation of the $\ell=1$ states where
   $$|1,1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}\ ,\quad |1,0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}\ ,\quad |1,-1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}\ , \tag{2}$$
   use (1) to find matrix representations of $L_\pm$ and then L_x and L_y.

   (c) *Griffiths 4.22(b)* Use $L_+ \cdot Y_\ell^\ell = 0$ and $L_z \cdot Y_\ell^\ell = \ell\hbar Y_\ell^\ell$ to determine $Y_\ell^\ell(\theta,\phi)$ up to overall normalization.

2. **Commutators and Things** *partly from Griffiths 4.19, partly inspired by problems in Ohanian*

   (a) Find the commutators $[L_z, x]$, $[L_z, y]$, $[L_z, p_x]$, and $[L_z, p_y]$.

   (b) Show that $[L_z, H] = 0$ for Hamiltonian $H = \vec{p}^2/2m + V$ with central potential V. Argue that therefore $[\vec{L}^2, H] = 0$ also. *Hint:* Assume $V(r)$ is a power series; then argue L_z and r commute based on $[L_z, r^2]$.

   (c) Using the commutation relations, find $\vec{L}\times\vec{L}$ (as an operator quantity). How does your result relate to the classical statement that a vector cross itself is zero?

3. **Probabilities and Expectations** *based on problems by Ohanian*

   Some particle has a wavefunction
   $$\psi(\vec{x}) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\sin^2\theta\left[1+\sqrt{14}\cos\theta\right]\cos(2\phi)R(r)\ , \tag{3}$$
   where R is normalized ($\int_0^\infty dr\, r^2|R|^2 = 1$).

   (a) If you measure the total orbital angular momentum $\vec{L}^2$, what are the possible values you would find and the probabilities you would measure each value?

   (b) Find $\langle L_x\rangle$, $\langle L_y\rangle$, and $\langle L_z\rangle$ in this state.

   (c) Find the uncertainty of L_z.

4. **Angular Dependence of Spherical Harmonic Oscillator**

   Consider a 3D isotropic harmonic oscillator (potential $V(\vec{x}) = m\omega^2 r^2/2$). Recall from a previous assignment that separation of variables in Cartesian coordinates allows us to write the eigenstates as $|n_x,n_y,n_z\rangle = (1/\sqrt{n_x!n_y!n_z!})(a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}(a_z^\dagger)^{n_z}|0\rangle$ in terms of the Cartesian harmonic oscillator ladder operators ($a_x, a_x^\dagger$, etc). Write L_z in terms of the Cartesian harmonic oscillator ladder operators. Then use ladder operator techniques to show that $(|1,0,0\rangle \pm i|0,1,0\rangle)/\sqrt{2}$ and $|0,0,1\rangle$ are eigenstates of L_z and find their eigenvalues.

5. **Selection Rules and Degeneracy**

   The angular momentum commutation relations can give us information about matrix elements ("selection rules") and eigenvalues (degeneracy). Let $|n, \ell, m\rangle$ and $|n, \ell, m'\rangle$ be $\vec{L}^2, L_z$ eigenstates labeled by ℓ, m (or m') as usual, where n is an additional quantum number describing the radial wavefunction.

   (a) Using the commutation relations, prove that $\langle n, \ell, m'|L_x|n, \ell, m\rangle = i(m'-m)\langle n, \ell, m'|L_y|n, \ell, m\rangle$.

   (b) In problem 2(b), we saw that the Hamiltonian with a central potential V commutes with all the angular momentum operators. Use this fact to show that, if $|n, \ell, m'\rangle$ is an energy eigenstate with eigenvalue E, the states $|n, \ell, m\rangle$ with the same n, ℓ are all energy eigenstates for the same energy for all allowed values of m given a central potential.

6. **General Spin Components**

   In this problem, you will consider a more general component of spin (rather than one along x, y, or z). In the below, define a unit vector $\hat{a} = \cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}$. Assume the total spin is $s = 1/2$.

   (a) Find the eigenvectors of $\hat{a} \cdot \vec{S}$ in the eigenbasis of S_z.

   (b) Suppose an electron in the eigenstate of $\hat{a} \cdot \vec{S}$ with eigenvalue $+\hbar/2$ is placed in a magnetic field $\vec{B} = B_0\hat{z}$ direction. The Hamiltonian is $H = -\gamma\vec{B} \cdot \vec{S} = -\gamma B_0 S_z$ (this is a magnetic moment interaction). What is the probability that a measurement of the energy returns $E = -\gamma B_0\hbar/2$?

   (c) Calculate the expectation value of $S_x S_z S_x$ in the eigenstate of $\hat{a} \cdot \vec{S}$ with eigenvalue $+\hbar/2$.