PHYS-4601 Homework 10 Due 29 Nov 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Particle in a Box and Degeneracy** *Based on Griffiths 4.2*

   Consider a 3D square well with potential

   $$V(x,y,z) = \begin{cases} 0 & 0 < x,y,z < a \\ \infty & \text{otherwise} \end{cases} . \tag{1}$$

   That is, the particle moves freely within a box with walls at $x = 0, a$, $y = 0, a$, and $z = 0, a$.

   (a) Find the wavefunctions and energies of the stationary states.

   (b) In 1D quantum mechanics, there is only one bound state for a given energy. In 3D, there can be more than one; we call stationary states with the same energy *degenerate*, and the number of states with a given energy is the *degeneracy*. Give the three lowest energy eigenvalues and their degeneracies.

   (c) Write the three lowest energy eigenvalues for a similar potential but with walls at $x = 0, 2a$, $y = 0, a$, and $z = 0, a$. What are the degeneracies?

2. **Isotropic Harmonic Oscillator** *from Griffiths 4.38,39*

   Consider a harmonic oscillator where the restoring force is independent of the direction. In this case, the potential is

   $$V(r) = \frac{1}{2} m\omega^2 r^2 . \tag{2}$$

   (a) Show that the energy eigenvalues are $E_n = \hbar\omega(n + 3/2)$, where n is any non-negative integer. It's easiest to do this using separation of variables in Cartesian coordinates.

   (b) Find the degeneracy of states with energy E_n.

   (c) Now consider the Schrödinger equation in spherical coordinates and restrict to the case $\ell = 0$ for the radial equation. We know that the ground state has $\psi \propto \exp[-\rho^2/2]$ from separation of variables in Cartesian coordinates, where $\rho = \sqrt{m\omega/\hbar}\, r$. Therefore, we define $u = v(\rho)\exp[-\rho^2/2]$, where $v(\rho) = \rho + \cdots$ is a polynomial. Argue that the (unnormalized) wavefunctions $u = H_n(\rho)\exp[-\rho^2/2]$ solve the radial equation for any odd n. Find the associated energies for $n = 1, 3$. Did you find the energies you expected from part (a)? Explain why or why not.

3. **Spherical Shell Well**

   Consider a central potential

   $$V(r) = \begin{cases} \infty & 0 < r < a \ ,\ 2a < r \\ 0 & a < r < 2a \end{cases} \tag{3}$$

   in three dimensions. For this potential, we do not need to worry about boundary conditions at the origin.

   (a) Find the energy eigenvalues for the $l = 0$ states.

(b) Using the boundary condition at $r = a$, argue that the radial wavefunction for general l can be written as $R(r) = A[n_l(ka)j_l(kr) - j_l(ka)n_l(kr)]$, where j_l, n_l are spherical Bessel functions and A is a normalization constant. Show therefore that the energy eigenvalues are given by the solutions $x_{n,l}$ of the equation $n_l(x_{n,l})j_l(2x_{n,l}) = j_l(x_{n,l})n_l(2x_{n,l})$.

(c) Use the information in the Digital Library of Mathematical Functions at `https://dlmf.nist.gov/10.47` to show that the energy eigenvalues can be expressed in terms of the solutions to $J_{-l-1/2}(x_{n,l})J_{l+1/2}(2x_{n,l}) = J_{l+1/2}(x_{n,l})J_{-l-1/2}(2x_{n,l})$, where $J_\nu(x)$ are Bessel functions (not spherical Bessel functions). (Note that the DLMF calls $n_l = y_l$.) Then use the maple command `fsolve(BesselJ(3/2, 2*x)*BesselJ(-3/2, x) = BesselJ(3/2, x)*BesselJ(-3/2, 2*x), x = 0..4)` to find the lowest energy eigenvalue for $l = 1$. You do not need to attach a printout of your maple work, just give the solution as found and the corresponding energy eigenvalue. Show your answers to 2 decimal places.