PHYS-4601 Homework 1 Due 13 Sept 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dirac Notation on the Circle**

   Consider the Hilbert space of L^2 functions on the interval $0 \leq x \leq 2\pi R$ with periodic boundary conditions.

   (a) Show that the complex exponentials $|e_n\rangle \simeq e^{inx/R}/\sqrt{2\pi R}$ for n any integer form an orthonormal set. As it turns out, they make a complete orthonormal basis (but you do not have to prove that).

   Carry out the following calculations *without doing any integrals*.

   (b) Calculate the inner product of $|f\rangle \simeq f(x) = \cos^3(x/R)$ and $|g\rangle \simeq g(x) = \sin(3x/R)$.

   (c) Find the inner product of $|f\rangle$ and $|g\rangle$ from part (b) with $|h\rangle \simeq h(x) = \sin(3x/R + \theta)$.

2. **Dual Vectors and Change of Basis**

   Consider a 3-dimensional complex column vector space, which has the usual orthonormal basis

$$|e_1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \ |e_2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \ |e_3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{1}$$

   In that basis, the vectors $|f_i\rangle$ $(i = 1, 2, 3)$ can be written as

$$|f_1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \ |f_2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad , \ |f_3\rangle \simeq \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ -i \\ -2i \end{bmatrix} . \tag{2}$$

   (a) Write the $|f_i\rangle$ as linear superpositions of the $|e_i\rangle$ basis vectors.

   (b) Show that the $|f_i\rangle$ are normalized and mutually orthogonal, so they form a complete orthonormal basis (distinct from the set of $|e_i\rangle$).

   (c) Write the associated dual vectors $\langle f_i|$ as row vectors in the $\{\langle e_i|\}$ basis.

   (d) Write the $|e_i\rangle$ vectors as linear superpositions of the $|f_i\rangle$. Use your result to do a change of basis for this Hilbert space by writing the $|e_i\rangle$ vectors as column vectors in the $\{|f_i\rangle\}$ basis. *Hint:* You can solve a system of linear equations or use a similarity transformation, but it is much easier if you use inner products as discussed in the notes.

3. **Superposition of States**

   Suppose $|\psi\rangle$ and $|\phi\rangle$ are two normalized state vectors, and so is $|\alpha\rangle = A(3|\psi\rangle + 4|\phi\rangle)$.

   (a) Find the normalization constant A in the case that

      i. $\langle\psi|\phi\rangle = 0$.

      ii. $\langle\psi|\phi\rangle = i$.

      iii. $\langle\psi|\phi\rangle = e^{i\pi/6}$.

(b) Now suppose that $\langle\psi|\phi\rangle = 0$ and define a new state $|\beta\rangle = B(4e^{-i\theta}|\psi\rangle + 3e^{i\theta}|\phi\rangle)$ for some angle θ. Find the normalization constant B and $\langle\alpha|\beta\rangle$ (you make assume that the normalization constants are positive and real).

4. **Gram-Schmidt Procedure**

Consider an N-dimensional vector space with inner product. Suppose we have a set $\{|e_1\rangle, |e_2\rangle, \cdots, |e_n\rangle\}$ of orthonormal vectors with $n < N$, so there are not enough of them to form a basis.

(a) Let $|\psi\rangle$ be linearly independent of the $|e_i\rangle$ (in other words, it is not a linear superposition of the $|e_i\rangle$ vectors). Show that

$$|\psi'\rangle \equiv |\psi\rangle - \sum_{i=1}^{n}(\langle e_i|\psi\rangle)|e_i\rangle \tag{3}$$

is orthogonal to all the $|e_i\rangle$ vectors. Then we can define the normalized vector $|e_{n+1}\rangle \equiv |\psi'\rangle/\sqrt{\langle\psi'|\psi'\rangle}$ as a new member of the orthonormal set $\{|e_i\rangle\}$. As a result, given enough linearly independent vectors, we can keep adding to the orthonormal set until we have enough to form an orthonormal basis. This is known as the *Gram-Schmidt procedure.*

(b) Consider the following vectors in a 3-dimensional Hilbert space

$$|\alpha\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad |\beta\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad |\gamma\rangle \simeq \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}. \tag{4}$$

Use the Gram-Schmidt procedure to write an orthonormal basis, as follows:

i. Normalize $|\alpha\rangle$ to find $|e_1\rangle$.
ii. With $n = 1$, find $|e_2\rangle$ by applying the procedure of part (a) to $|\beta\rangle$.
iii. Then apply the Gram-Schmidt procedure to $|\gamma\rangle$ using the two orthonormal vectors you found previously to find $|e_3\rangle$.

(c) If you instead first used $|\gamma\rangle$ to define $|e_1\rangle$, then $|\beta\rangle$ to find $|e_2\rangle$, and finally $|\alpha\rangle$ to find $|e_3\rangle$, would you have the same basis or not? Give your reasoning, but you do not need to carry out a full calculation.