– Electromagnetism + Charged Particles in QM

We will take a brief aside to look at the Schrödinger equation / Hamiltonian when EM is involved, including the effects on charged spins

• The Hamiltonian for Charged Particle Motion

+ The Lorentz force $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$ involves velocity, so it is not the gradient of a potential. We must modify our usual Hamiltonian

+ Remember: to describe given EM fields $\vec{E}$ + $\vec{B}$, we need two potentials

  Φ = (scalar) potential , $\vec{A}$ = vector potential

  with $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla}\times\vec{A}$

+ With no $\vec{B}$ field, we can just set $\vec{A}=0$ and $V(\vec{x}) = q\Phi(\vec{x})$

  With $\vec{B}$, $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi$$ for particle of charge q.

  Notes: 1) $\vec{p} = -i\hbar\vec{\nabla}$ as usual, but it is not $m\vec{v}$ (even classically)

  2) Usually, $\vec{p}$ and $\vec{A}$ commute, but be careful.

+ Ehrenfest's theorem lets you derive the Lorentz force law for expectation values

+ Classically, a gauge transformation $\Phi \to \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$ and $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$ leaves $\vec{E}$ + $\vec{B}$ unchanged.

  You can show that these plus the wavefunction change $\Psi \to \Psi' = e^{iq\Lambda/\hbar}\,\Psi$ leaves the (time-dependent) Schr. eqn unchanged.

- Aharonov-Bohm effect; weird effects of $\vec{B}$-fields

+ Suppose $\vec{B} \neq 0$ somewhere. Then $\vec{A}$ is generally $\neq 0$ even where $\vec{B}=0$.

  Ex There is a solenoid on the z axis that is very small + tightly wound,
  so $\vec{B} = B\hat{z}$ on the axis and $\vec{B}=0$ everywhere else.
  Then $\vec{A} = \frac{AB}{2\pi r}\hat{\phi}$ where A = cross-sectional area of solenoid, and $r, \hat{\phi}$ are cylindrical coordinates.

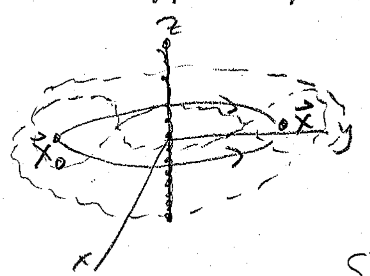
  This has $\vec{\nabla} \times \vec{A} = 0$ everywhere except $r=0$.

+ We can use a trick to solve the Schr. eqn.

  Define
  $$g(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} \vec{A}(\vec{x}')\cdot d\vec{x}' \Rightarrow \vec{\nabla} g = \vec{A}, \quad \vec{x}_0 = \text{arbitrary reference}$$

  In a region where $\vec{\nabla}\times\vec{A}=0$, this is path-independent, so we can define
  $$\Psi(\vec{x},t) = e^{iqg(\vec{x})/\hbar}\,\Psi'(\vec{x},t) \qquad (*)$$
  Then (you check!) $(-i\hbar\vec{\nabla} - q\vec{A})\Psi = -i\hbar\, e^{iqg/\hbar}\,\vec{\nabla}\Psi'$, so
  $$-\frac{\hbar^2}{2m}\nabla^2\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}$$

+ Now suppose you have that example solenoid.



  Then $\vec{B} \neq 0$ on the z-axis.

  You can only use $(*)$ in regions not containing the z-axis.

  So semicircular paths from $\vec{x}_0 = -a\hat{y}$ to $\vec{x} = +a\hat{y}$ in the + or − x region have different phases
  $$\frac{qg}{\hbar} = \frac{q}{\hbar}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{AB}{2\pi a}\hat{\phi}\cdot(\pm a\, d\phi\,\hat{\phi}) = \pm\frac{qAB}{2\hbar}.$$
  In other words, Ψ is not single-valued.

+ This is the Aharonov-Bohm effect. There is a <u>measurable</u> interference due to $\vec{B}$ even when $\Psi = 0$ where $\vec{B} \neq 0$!

- Magnetic Fields + Spins

+ We've seen on homework how electromagnetic fields affect the Hamiltonian of a charged particle.

But there is *another* very important effect:

A spinning charged particle has a magnetic dipole moment $\vec{\mu} = \gamma \vec{S}$

γ = gyromagnetic ratio $\simeq -e/m$ for electron

From E+M, a dipole in magnetic field has Hamiltonian

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma \vec{S}\cdot\vec{B}$$

+ An example: Larmor precession

Put an electron at rest in $\vec{B} = B_0\hat{z}$. $H = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

Eigenstates + energy eigenvalues

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; E_+ = -\gamma B_0\hbar/2 \; , \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_- = \gamma B_0 \hbar/2$$

An initial state $|\Psi\rangle = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix} \longrightarrow |\Psi(t)\rangle = \begin{bmatrix} \cos(\alpha/2)\, e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\, e^{-i\gamma B_0 t/2} \end{bmatrix}$

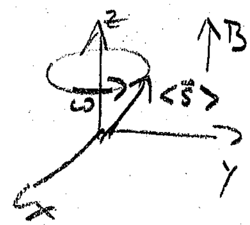
As shown in the book, the expectation value is matrix multiplication

$$\langle S_z \rangle = \frac{\hbar}{2}\begin{bmatrix} a^* & b^* \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \hbar/2\left(|a|^2 - |b|^2\right) \rightarrow \frac{\hbar}{2}\cos\alpha \quad \text{constant}$$

But

$$\langle S_x \rangle = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t), \quad \langle S_y \rangle = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t)$$

So $\langle \vec{S} \rangle$ precesses at Larmor frequency $\omega = \gamma B_0$ around $\vec{B}$.

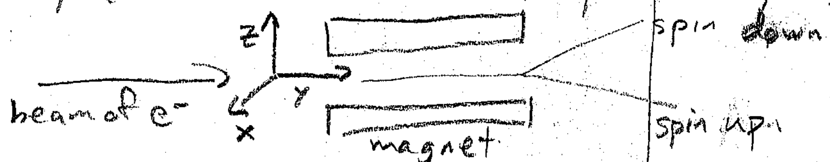


Note: orientation b/c γ is negative

+ Stern-Gerlach experiment a famous historical example

If $\vec{B}$ is non-uniform, H depends on position $\Rightarrow$ force.

But sign/direction of force depends on spin. You can separate spins

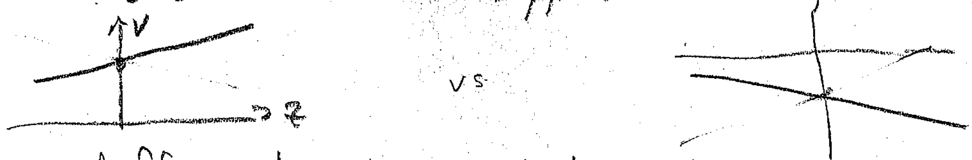


In magnet,

$\vec{B} = (B_0 + \alpha z)\hat{z} - \alpha x \hat{x}$

1) The x-component is needed for Maxwell's eqns but unimportant b/c Larmor precession averages it away for large B_0

2) Potential in z direction looks opposite for 2 spins



Generates different p_z as electron traverses magnet

3) Actual calculation a more complicated 3D scattering problem

Book gives heuristic calculation for a state $|\Psi\rangle = |\Psi_+\rangle|+\rangle + |\Psi_-\rangle|-\rangle$

Let's consider states $|+\rangle$ and $|-\rangle$ separately and look at $\langle p_z \rangle$

By Ehrenfest,

$$\frac{d\langle p_z\rangle}{dt} = \frac{i}{\hbar}\langle [H, p_z]\rangle = \frac{i}{\hbar}(-\gamma\alpha)\langle \pm|S_z|\pm\rangle\langle\Psi_\pm|[z,p_z]|\Psi_\pm\rangle$$

$$= \pm\gamma\alpha\hbar/2$$

4) So this is a method to measure spins of individual particles

Allows us to prove existence of spin 1/2 ← integer spins separate into odd # of spots on screen, half-integer into even.