PHYS-3301 Winter Homework 9 Due 21 Mar 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Some Short Practice Calculations**

   Calculate the following quantities. You should get a number for each answer.

   (a) $\eta_{\mu\nu}\eta^{\mu\nu}$

   (b) $\eta^{\mu\nu}\eta^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$

   (c) $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\nu\lambda\rho}$

   Based on a problem by Carroll In the next two calculations, define the tensor $X^{\mu\nu}$ and vector V^μ by

$$\left[\begin{array}{c} \\ X^{\mu\nu} \\ \\ \end{array}\right] = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix} \ , \quad V^\mu = (-1, 2, 0, -2) \tag{1}$$

   in some inertial frame S. Then calculate the following:

   (d) $X^\mu{}_\mu$ (This is called the *trace* of X.)

   (e) $X^{\mu\nu}V_\mu V_\nu$

2. **The Relativistic Electromagnetic Field**

   We won't prove it, but the electric and magnetic fields can be written as a relativistic tensor with two indices $F^{\mu\nu}$. This tensor is *antisymmetric*, meaning $F^{\nu\mu} = -F^{\mu\nu}$. The independent components are (here, $i = 1, 2, 3$ is a space index)

$$F^{0i} = E^i \ , \quad F^{12} = B^3 \ , \quad F^{13} = -B^2 \ , \quad F^{23} = B^1 \ . \tag{2}$$

   Since $F^{\mu\nu}$ is antisymmetric, the diagonal components $F^{00} = F^{11} = F^{22} = F^{33} = 0$. (We have chosen a convenient system of units where the electric and magnetic field have the same dimension.)

   (a) Consider two frames S and S' in standard configuration with each other. Show that

$$E^{3'} = \gamma\left(E^3 + \frac{v}{c}B^2\right) \quad \text{and} \quad B^{3'} = \gamma\left(B^3 - \frac{v}{c}E^2\right) \ . \tag{3}$$

   Hint: Remember that the Lorentz transformation of a tensor transforms each index independently:

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta F^{\alpha\beta} \ . \tag{4}$$

   (b) Calculate $F_{\mu\nu}F^{\mu\nu}$ and argue that $\vec{E}^2 - \vec{B}^2$ is a Lorentz invariant quantity.

3. **Lorentz Force**

   The Lorentz force is the force on a moving charge due to electric and magnetic fields. In relativistic covariant notation, the Lorentz force is

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}U_\nu F^{\mu\nu} \ , \tag{5}$$

   where p^μ is the particle's momentum, q the charge, U^μ the particle's 4-velocity, and $F^{\mu\nu}$ the electromagnetic field defined as in equation (2).

(a) Show that the $\mu = 1$ component of equation (5) reproduces the Lorentz force law in the x direction

$$\frac{dp^1}{dt} = qE^1 + \frac{qu^2}{c}B^3 - \frac{qu^3}{c}B^2 \ , \tag{6}$$

where u^i is the coordinate velocity.

(b) Express the $\mu = 0$ component of equation (5) in terms of the power dE/dt delivered to the particle, its coordinate velocity, and the $\vec{E}$ and $\vec{B}$ fields. What is the physical interpretation of this equation?