PHYS-3301 Winter Homework 8 Due 14 Mar 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

$$a^{\mu'} = \Lambda^{\mu'}{}_{\nu} a^{\nu} \quad \text{and} \quad a_{\mu'} = \Lambda^{\nu}{}_{\mu'} a_{\nu} \ , \tag{1}$$

   where $[\Lambda^{\mu'}{}_{\nu}]$ is the usual Lorentz transformation matrix from $S \to S'$ and $[\Lambda^{\nu}{}_{\mu'}]$ is its matrix inverse (the transformation from $S' \to S$).

   (a) Using the fact that the spacetime position x^μ is a 4-vector, find the partial derivatives $\partial x^\mu / \partial x^{\nu'}$ and $\partial x^{\mu'} / \partial x^\nu$ in terms of $\Lambda^{\mu'}{}_{\nu}$ and $\Lambda^{\nu}{}_{\mu'}$. *Hint:* For two positions as measured in the same frame, $\partial x^\mu / \partial x^\nu = \delta^\mu_\nu$ (think about why).

   (b) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the multivariable chain rule to show that

$$\frac{\partial f}{\partial x^{\mu'}} = \Lambda^{\nu}{}_{\mu'} \frac{\partial f}{\partial x^{\nu}} \ . \tag{2}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_\mu f \equiv \partial f / \partial x^\mu$.

2. **4-Vectors and Changing Frames** Consider a 4-vector U^μ with components $U^0 = \cosh\phi$, $U^1 = \sinh\phi$, $U^2 = U^3 = 0$ in some reference frame S. *Hint:* You will want to remember hyperbolic trig identities.

   (a) What is the square of U^μ? Is U^μ lightlike, timelike, or spacelike?

   (b) Now boost to reference frame S' which has velocity $v = c\tanh\theta$ along the x direction relative to S. What are the components of U^μ in S'? Write your answer in terms of the "angle" $\phi - \theta$.

   (c) If $\phi - \theta$ is itself a rapidity, write the associated velocity in terms of $\tanh\theta$ and $\tanh\phi$.

   (d) Instead, consider a boost to a frame S'' with velocity $v = c\tanh\theta$ along the y direction relative to S. This has Lorentz transformation matrix

$$\left[\Lambda^{\mu''}{}_{\nu}\right] = \begin{bmatrix} \cosh\theta & 0 & -\sinh\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ . \tag{3}$$

   What are the components of U^μ in frame S''?

   (e) Find the square U^2 as calculated in the S'' frame using your answer to part (d). Is it the same as your answer from part (a)?

3. **Some Scalar Products**

   In some frame, the components of two 4-vectors are

$$a^\mu = (2, 0, 0, 1) \text{ and } b^\mu = (5, 4, 3, 0) \ . \tag{4}$$

   inspired by a problem in Hartle

(a) Find a^2, b^2, and $a \cdot b$.

(b) Does there exist another inertial frame in which the components of a^μ are $(1, 0, 0, 1)$? What about b^μ? Explain your reasoning.

Now consider lightlike 4-vectors f^μ and g^μ.

(c) If f^μ and g^μ are orthogonal ($f \cdot g = 0$), prove that they are parallel ($f^\mu \propto g^\mu$).

(d) Is the 4-vector $f^\mu + g^\mu$ spacelike, timelike, or lightlike? Assume that both $f^0 > 0$ and $g^0 > 0$.

4. **Energy-Momentum Tensor**

The energy-momentum tensor $T^{\mu\nu}$ describes the energy and momentum densities of a fluid. For a fluid with no shear stress, the components are defined as $T^{00} = \rho$, the energy density; $T^{0i} = T^{i0} = \mathcal{P}^i$, the density of the ith component of momentum; and $T^{ij} = p\delta^{ij}$ is the pressure (we assume that the pressure is the same in all directions).

(a) List all the components $T_{\mu\nu}$ with lowered indices.

(b) In the fluid's rest frame S, $\mathcal{P}^i = 0$. What is the momentum density in a frame S' moving at speed v in the $+x$ direction with respect to S if the fluid is radiation with $p = \rho/3$?

(c) If the fluid is vacuum energy, the energy-momentum tensor has $\mathcal{P}^i = 0$ and $p = -\rho$ in its rest frame S. Show that the energy-momentum tensor is the same in all inertial frames. That is, given another frame S' moving at speed v in the $+x$ direction with respect to S, show that $[T^{\mu'\nu'}] = [T^{\mu\nu}]$ as a matrix.