PHYS-3301 Winter Homework 3 Due 31 Jan 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Rotations and Orthogonal Matrices**

   Two position vectors have the dot product $\vec{x} \cdot \vec{y} = x^i y^i$ (remember, superscripts tell you the component, and we use Einstein summation convention). The length of a vector $|\vec{x}|$ is given by the vector's dot product with itself as follows: $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$.

   (a) Rotations leave the lengths of vectors invariant (unchanged). Prove that this means that dot products are invariant under rotations.

   (b) Use the invariance of all dot products under rotations to show that all rotation matrices R satisfy the identity

   $$R^i{}_j R^i{}_k = \delta_{jk} \ , \tag{1}$$

   where δ_{jk} is the Kronecker delta symbol (equal to 1 if $j = k$ and 0 otherwise). *Hint:* To prove (1), consider frame S' rotated by R with respect to frame S. Then write the dot product $\vec{x}' \cdot \vec{y}'$ in terms of $\vec{x}, \vec{y}$ and set it equal to $\vec{x} \cdot \vec{y}$. You get an equation true for all $\vec{x}, \vec{y}$, which allows you to cancel the vector components from both sides of the equation.

   (c) Treat each column of R as a vector. Show that all the columns are perpendicular to each other and have length 1. This means that rotation matrices are *orthogonal matrices*.

2. **Rotating Frame**

   A tennis ball has position $\vec{x}$ and constant velocity $\vec{v}$ in inertial frame S. Frame S' is a *noninertial* frame that rotates around the z axis with angular velocity ω (that is, the angle it is rotated compared to S is $\theta = \omega t$). In frame S', the tennis ball is accelerating, so we invent *fictitious forces* just so we can say $\vec{F}_{fict} = m\vec{a}'$ in frame S', where m is the mass of the ball. *Hint:* you will want to recall the example rotation matrix from the class notes.

   (a) Consider the case where the tennis ball is stationary with position $\vec{x} = (x, 0, 0)^T$ in frame S. First, show that the ball appears to have circular motion in frame S'. Then show that the fictitious force is directed toward the origin with the correct magnitude to account for that circular motion.

   (b) Next, suppose the ball is initially located at $\vec{x} = (x, 0, 0)^T$ with velocity $\vec{v} = (v, 0, 0)^T$ in frame S (outward directed motion). What is the fictitious force felt by the ball at $t = 0$? This includes the *Coriolis effect*.

3. **Choosing Frames Wisely**

   In each part, clearly state what inertial reference frame you use to solve the problem.

   (a) An airplane flies at a constant velocity toward downtown Winnipeg. At some point, the pilot drops a care package of chocolate bars, which lands in the lawn in front of Wesley Hall. In which direction should the pilot look to see the happy faces on all the chocoholic students when the package lands? Explain, and ignore air resistance.

   (b) *Barton 2.2 rephrased* A river flows at 5 km/hr, and a boat in it can move 8 km/hr relative to the water. As the boat moves upstream, the driver hears a splash but only realizes that it was the life preserver falling overboard 15 minutes later. The driver turns around and heads to retrieve the life preserver. How soon can the boat catch up to the life preserver?

4. **Parallel Axis Theorem**

The *moment of inertia* of a collection of particles around a given axis is $I \equiv \sum_i m_i r_i^2$, where r_i is the distance of the individual particle from the axis of rotation (ie, if the axis of rotation is the z axis, r_i is the distance in the x, y plane). For a continuous object, replace the sum by an integral overinfinitesimal mass elements dm.

Show that the moment of inertia around an axis a perpendicular distance h away from the object's center of mass is $I = I_{CM} + Mh^2$, where M is the total mass and I_{CM} is the moment of inertia around a parallel axis running through the center of mass, as in the figure.