PHYS-3301 Winter Homework 12

This homework is **NOT TO BE HANDED IN OR GRADED**. It is for studying and preparation for the final exam. Solutions will be posted on April 9.

1. **The Cosmic Microwave Background**

   The universe is filled with left-over radiation from the Big Bang called the Cosmic Microwave Background, which has a typical frequency of $\bar{\omega} = 160$ GHz averaged over the sky. There is a smooth variation of frequency at different points on the sky. The maximum observed frequency is 200 MHz higher than average in one particular spot on the sky; the minimum frequency is 200 MHz lower than average at the diametrically opposite point. This effect is due to the motion of our galaxy relative to the overall rest frame of the microwave background.

   (a) What is the relative speed of the earth and the source of the Cosmic Microwave Background in km/s? Remember that $c \approx 3 \times 10^8$ m/s.

   (b) Suppose we measure the frequency at an angle θ on the sky from the maximum frequency point. Show that the frequency is

   $$\omega(\theta) = \bar{\omega} + (200 \text{ MHz}) \cos\theta \ . \qquad (1)$$

2. **Doppler vs Time Dilation**

   A starship travels directly away from earth. According to an observer on the ship, a red light (angular frequency $\omega = 3000$ THz) on the tail of the ship blinks once per second.

   (a) An observer on the earth sees the light blink once per every 3 seconds. At that point, what is the angular frequency of the light as observed on earth?

   (b) How often does the light on the ship blink measured by the clocks of the earth's rest frame (not an observer at a fixed location)?

3. **GZK Cut-off**

   The universe is filled with photons left over from the Big Bang which have a typical energy 2.5×10^{-4} eV. This is called the cosmic microwave background (CMB) radiation. Suppose a proton of energy E and mass m_p hits a typical CMB photon head-on. What is the minimum value of E required to convert the photon to a pion of mass m_π? In other words, how large must E be for the reaction $p + \gamma \to p + \pi^0$ to occur? Use $m_p = 938$ MeV$/c^2$ and $m_\pi = 135$ MeV$/c^2$ and assume that $m_p \ll E$. FYI, Greisen, Zatsepin, and Kuzmin predicted that protons of higher energies cannot travel for extremely long distances through the universe because they lose energy to pion production in this way.

4. **Relativistic Newton's Cradle**

   A particle of type B smashes into a particle of type C, turning into one particle of type A and one of type D, with masses $m_A < m_B < m_C < m_D$. Suppose that the C particle is intially at rest and the D particle is produced also at rest. (This can really happen when the B particle is a kaon, the C is a proton, the D is a lambda hyperon, and the A is a pion.)

   (a) What is the initial energy of the B particle?

   (b) What is the final energy of the A particle?

5. **4-Velocity Relations**

   (a) Consider an object moving with 4-velocity U^μ. Find the time component U^0 of the 4-velocity and use it to show that the relativistic gamma factor can be written as $\gamma = \sqrt{1 + \vec{U}^2/c^2}$.

   (b) Define the 4-acceleration of the object as the 4-vector $A^\mu = dU^\mu/d\tau$. Prove that $A_\mu U^\mu = 0$.

   (c) If two particles move with 4-velocities U_1^μ and U_2^μ with respect to some laboratory frame, show that the scalar product $U_1^\mu U_{2,\mu}$ is equal to $-c^2\gamma$, where γ is the relativistic gamma factor of the second particle's velocity relative to the first particle.

6. **Trace of Energy-Momentum**

   The energy-momentum tensor $T^{\mu\nu}$ describes the energy and momentum densities of a fluid. The components are defined as $T^{00} = \rho$, the energy density; $T^{0i} = T^{i0} = \mathcal{P}^i$, the density of the ith component of momentum; $T^{ij} = \sigma^{ij}$ for $i \neq j$ describes the shear stress; and T^{ij} with $i = j$ is the pressure P (we assume that the pressure is the same in all directions). Calculate $\eta_{\mu\nu}T^{\mu\nu}$.

7. **4-Vector Arithmetic**

   Suppose the components of a^μ in frame S are $a^0 = k$, $a^1 = k$, and $a^2 = a^3 = 0$.

   (a) Is a^μ timelike, spacelike, or lightlike?

   (b) If frame S' is related to S by a boost of velocity v along the x direction, show that the components in frame S' are $a^{0'} = k'$, $a^{1'} = k'$, and $a^{2'} = a^{3'} = 0$, where $k' = \sqrt{(1 - v/c)/(1 + v/c)}\, k$.

   (c) Find the components of the 4-vector $b^\nu = a_\mu T^{\mu\nu}$ and the value of the scalar $a_\mu a_\nu T^{\mu\nu}$, where $T^{\mu\nu}$ is the energy-momentum tensor defined in problem 6 above (assume it is given in frame S).