PHYS-3301 Winter Homework 1 Due 17 Jan 2018

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

Helpful Formulae: The following may be useful:

$$\int_0^\infty dx\, x^n e^{-x/a} = n!\, a^{n+1} \quad (a > 0,\ n = 0, 1, 2, \cdots) \tag{1}$$

$$\int_{-\infty}^\infty \frac{dk}{2\pi}\, e^{ik(x-x')} = \delta(x - x') \tag{2}$$

$$\int_{-\infty}^\infty dx\, x^{2n} e^{-x^2/b^2} = 2\sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{b}{2}\right)^{2n+1} \quad (\mathrm{Re}\, b > 0)\ . \tag{3}$$

1. **Momentum Space Wavefunctions**

   For a particle moving on the real line ($-\infty < x < \infty$) with wavefunction $\psi(x)$, the "momentum space wavefunction" is

   $$\tilde{\psi}(p) = \int_{-\infty}^\infty dx\, \psi_p(x)^* \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty dx\, e^{-ipx/\hbar}\psi(x)\ . \tag{4}$$

   (a) A particle has normalized wavefunction $\psi(x) = \sqrt{\kappa} e^{-\kappa|x|}$ for some constant κ. Find the momentum space wavefunction and the momentum that maximizes $|\tilde{\psi}(p)|^2$ (this is the most likely momentum to measure). Sketch $\psi(x)$ and $\tilde{\psi}(p)$. What is the characteristic width of each wavefunction?

   (b) Suppose instead that the wavefunction is a normalized Gaussian $\psi(x) = (2a/\pi)^{1/4} \exp(-ax^2)$. Find the momentum space wavefunction. *Hint:* Combine exponentials in equation (4) and complete the square in the exponent. Then shift integration variables to get a Gaussian integral.

   (c) The width of a Gaussian function of the form $A \exp(-x^2/b^2)$ (with A the normalization constant) is $b/2$. Use your results from part (b) to find the width Δx of the wavefunction $\psi(x)$ and the width Δp of the momentum space wavefunction $\tilde{\psi}(p)$. Show that they saturate the Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$.

2. **Position and Momentum Operators in Momentum Space**

   (a) If the position operator acts on a wavefunction $\psi(x)$, it returns the wavefunction $x\psi(x)$. Show that the momentum space wavefunction corresponding to this is $i\hbar d\tilde{\psi}/dp$. In other words, the wavefunction $\Psi(x) = x\psi(x)$ has momentum space wavefunction $\tilde{\Psi}(p) = i\hbar d\psi/dp$.

   (b) If the momentum operator acts on a wavefunction $\psi(x)$, it returns the wavefunction $-i\hbar d\psi/dx$. Show that the momentum space wavefunction corresponding to this is $p\tilde{\psi}(p)$.

3. **Wavefunction Normalization**

   Show that normalization of the momentum space wavefunction implies normalization of the position space wavefunction. That is, show

   $$\int_{-\infty}^\infty dp\, |\tilde{\psi}(p)|^2 = 1 \ \Rightarrow\ \int_{-\infty}^\infty dx\, |\psi(x)|^2 = 1\ . \tag{5}$$

Hint: Substitute two factors of $\tilde{\psi}$ from equation (4) using different dummy integration variables. Then use the definition of the delta function (2).

4. **Spherical Harmonics**

   You may want to consult table 4.3 in Griffiths.

   (a) Write the function $\cos(2\theta)$ in terms of spherical harmonics.

   (b) Is $\cos(2\theta)$ an eigenfunction of L_z or $\vec{L}^2$ (or both)? Give the eigenvalue for any operator for which it is an eigenfunction.

   (c) Find the probability that an electron in the $n = 3, \ell = 0, m = 0$ state of hydrogen is found within an angle $\pi/3$ of the north pole (for polar angle $\theta < \pi/3$). Then find the probability if the electron is in the $n = 2, \ell = 1, m = 1$ state. Remember that the radial part of the wavefunction is normalized separately.