Quantum Mechanics I PHYS-3301 Final

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Instructions:

- Do not turn over until instructed. You will have 3 hours to complete this exam.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS ABOUT THE QUESTIONS WILL GO HERE.
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Concepts & Formulae:

- Physical Constants
  - Speed of light $c = 3 \times 10^8$ m/s = 1 lightsecond/second = 1 lightyear/year
  - Planck constant $h = 2\pi\hbar = 7 \times 10^{-34}$ Js
  - Boltzmann constant $k_B = 10^{-23}$ J/K

- Quantum Mechanics
  - Spherical harmonics will be given if needed
  - Momentum space wavefunction $\tilde{\psi}(p)$ with $p \cdot \tilde{\psi} = p\tilde{\psi}$, $x \cdot \tilde{\psi} = i\hbar d\tilde{\psi}/dp$

$$\tilde{\psi}(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x) \ , \quad \psi(x) = \int \frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\tilde{\psi}(p)$$

  - Maxwell-Boltzmann, Fermi-Dirac, Bose-Einstein distributions

$$n_{MB}(E) = e^{-(E-\mu)/k_BT} \ , \quad n_{FD}(E) = \frac{1}{e^{(E-\mu)/k_BT}+1} \ , \quad n_{BE}(E) = \frac{1}{e^{(E-\mu)/k_BT}-1}$$

  - Degeneracy $d_k = (sV/2\pi^2)k^2dk$ where s =number of polarizations, $\vec{p} = \hbar\vec{k}$ (de Broglie)

- Galilean Relativity/Newtonian Mechanics
  - The CM frame is the frame in which the total spatial momentum is zero.
  - Rotations are given by multiplication with a rotation matrix.
  - Galilean boost $\vec{x}' = \vec{x} - \vec{v}t$, $\vec{u}' = \vec{u} - \vec{v}$, $\vec{p}' = \vec{p} - m\vec{v}$, $k' = k - \vec{p} \cdot \vec{v} + (1/2)mv^2$
  - Kinetic energy for many particles $K = K_{int} + (1/2)MV^2$

$$M = \sum m_i \ , \quad \vec{V} = \frac{1}{M}\sum m_i\vec{u}_i$$

  - For two particles $K_{int} = (1/2)\mu u^2$ for relative velocity $\vec{u}$ and reduced mass $\mu = m_1m_2/M$

- 4-vectors and Lorentz transformations
  - The position 4-vector is x^μ with $x^0 = ct$.
  - The metric $\eta_{\mu\nu}$ can be written as a diagonal matrix with diagonal elements $-1, 1, 1, 1$, and the invariant interval is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + d\vec{x}^2 = -c^2d\tau^2$.

- The Lorentz boost transformations (in standard configuration) are

$$t' = \gamma(t - vx/c^2)\ ,\ x' = \gamma(x - vt)\ ,\quad y' = y\ ,\quad z' = z\ ,\quad \gamma = 1/\sqrt{1 - v^2/c^2}\ .$$

  They can be written as $x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$
- Lowered indices $a_\mu = \eta_{\mu\nu} a^\nu$ (both in frame S)
- Relativistic dot product $a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = -a^0 b^0 + \vec{a} \cdot \vec{b}$
- Tensor transformation $T_{\mu'\cdots}{}^{\nu'\cdots} = (\Lambda^{\alpha}{}_{\mu'} \cdots)(\Lambda^{\nu'}{}_{\beta} \cdots) T_{\alpha\cdots}{}^{\beta\cdots}$

- Velocities and Momenta
  - For normal velocity $\vec{u} = d\vec{x}/dt$, the Lorentz transformation in standard configuration is

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}\ ,\quad u'_{y,z} = \frac{u_{y,z}}{\gamma(v)(1 - vu_x/c^2)}\ .$$

  - 4-velocity: $U^\mu = dx^\mu/d\tau$, where τ is proper time along the worldline. $U^0 = \gamma c$, $\vec{U} = \gamma d\vec{x}/dt$, $d\vec{x}/dt = c(\vec{U}/U^0)$.
  - 4-momentum is $p^\mu = mU^\mu$. Energy $E = cp^0$ and momentum is the spatial part $\vec{p}$.
  - $U_\mu U^\mu = -c^2$ and $p_\mu p^\mu = -m^2c^2$ for a normal massive particle.
  - Invariant mass $P_{tot,\mu} P^\mu_{tot} = -M^2c^2$, $M \geq \sum_n m_n$ for final state particle masses m_n

- The Doppler effect, in terms of the rest frame of the receiver, is

$$\frac{\omega_R}{\omega_E} = \frac{\sqrt{1 - u_E^2/c^2}}{1 - \hat{k} \cdot \vec{u}_E/c}\ ,$$

  where $\hat{k}$ is the direction of travel of light and $\vec{u}_E$ is the velocity of emitter relative to receiver.

- Math
  - Polar coordinates $d^3\vec{x} = r^2 \sin\theta\, dr\, d\theta\, d\phi$
    The region $-\infty < x, y, z < \infty$ is $0 < r < \infty, 0 < \theta < \pi, 0 < \phi < 2\pi$
  - Exponential integrals (for $n = 0, 1, \cdots$ and $a > 0$)

$$\int_0^\infty dx\, x^n e^{-x/a} = n! a^{n+1}$$

  - Gaussian integrals

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}\ ,\quad \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

  - Hyperbolic trig functions: $d\cosh\theta/d\theta = \sinh\theta$, $d\sinh\theta/d\theta = \cosh\theta$

$$\cosh^2\theta - \sinh^2\theta = 1\ ,\quad \cosh^2\theta + \sinh^2\theta = \cosh(2\theta)\ ,\quad 2\sinh\theta\cosh\theta = \sinh(2\theta)$$

  - Binomial expansion $(1 + x)^n \approx 1 + nx$ for $x \ll 1$