4-Vectors

- The velocity transformation rule is too complicated.
  (And we didn't even look at something like acceleration)
  - Why do we care? Remember, physical laws must be <u>covariant</u>:
    + The L.H.S. of an equation must have the same transformation as the RHS.
    + To check that, we need well-organized sets of variables with simple transformation rules
  - We've seen an analogous case: <u>Rotations</u>
    + Remember that <u>vectors</u> all have the same rotation transformation rule

$$x^{i'} = R^{i'}{}_j x^j$$ w/<u>Einstein summation convention</u>.

+ $x^{i'}$ = S' vector, x^j = S frame vector, $R^{i'}{}_j$ = rotation matrix

For example, for a z-axis rotation

$$[R^{i'}{}_j] = \begin{array}{c|c} & j \rightarrow \\ \hline i' \downarrow & \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Recall how the sum = matrix multiplication

+ It's easy to check if equations are covariant b/c we know what's a vector or a <u>scalar</u> (a rotational invariant)

  vectors: position, velocity, acceleration, momentum, force, ...

  Scalars: time, temperature, energy, mass, distance, ...

+ A couple of examples:

  Newton: $F^i = ma^i \rightarrow R^{i'}{}_j F^j = m R^{i'}{}_j a^j \rightarrow F^{i'} = ma^{i'}$ ✓

  Coulomb potential: $V(r) = q_1 q_2 / 4\pi\varepsilon_0 r$ ← all scalars. ✓

- Rotations are linear transformations on coordinates.
  So are the Lorentz transformations (boosts). We can make covariance under boosts easy to understand with this analogy.

- We can turn our coordinates into 4-vectors (index notation, this is standard, but the reading uses odd notation)

$$x^{\mu} = (ct, x, y, z) \quad \text{or} \quad x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

- We will use Greek indices $\mu, \nu, \ldots$ for all spacetime coordinates and Latin $i, j, \cdots$ for spatial directions only

- Lorentz boosts on 4-vectors
  - As in rotations, write a Lorentz transformation as matrix mult.

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \qquad \text{ie} \quad (S' \text{ frame}) = (\text{boost}) \cdot (S \text{ frame})$$

  - The boost can be represented as a matrix

$$\left[\Lambda^{\mu'}{}_{\nu}\right] = \begin{array}{c|c} & \nu \rightarrow \quad 0 \quad\ 1 \quad\ 2 \quad 3 \\ \hline \mu' \downarrow \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{bmatrix} \end{array}$$

Working out the sums, as usual:

$$ct' = \gamma(ct) - \left(\gamma \tfrac{v}{c}\right)(x), \quad x' = \gamma x - \left(\gamma \tfrac{v}{c}\right)(ct), \quad y' = y, \quad z' = z$$

  - The analogy with rotations goes farther:

$$\gamma^2 - \gamma^2 \frac{v^2}{c^2} = \frac{1}{(1 - v^2/c^2)}(1 - v^2/c^2) = 1 \quad \text{like} \quad \cosh^2\theta - \sinh^2\theta = 1$$

We can define <u>rapidity</u> θ with $\cosh\theta = \gamma$, $\sinh\theta = \gamma v/c$

Turns Λ into a "hyperbolic rotation"

- A 4-vector contains a normal vector $x^{\mu} = (x^0, x^i)$ and rotations fit inside a general 4D Lorentz transformation:

$$\Lambda^{0'}{}_{0} = 1, \quad \Lambda^{0'}{}_{i} = \Lambda^{i'}{}_{0} = 0, \quad \Lambda^{i'}{}_{j} = R^{i'}{}_{j}$$

gives

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \Rightarrow x^{0'} = x^0, \quad x^{i'} = R^{i'}{}_{j} x^j \qquad \text{(can you see this?)}$$

- Not every vector is a position. Everything that rotates like a position is a vector. Similarly, <u>anything that Lorentz transforms like a spacetime position is a 4-vector.</u>

— Scalar Products

- For rotations, the dot product turns 2 vectors into a scalar (so we call it a <u>scalar product</u>).
  + The definition is $a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$
  + The fact that this is invariant means
  $$a^{i'} b^{i'} = R^{i'}{}_j a^j R^{i'}{}_k b^k = a^j b^j \Rightarrow R^{i'}{}_j R^{i'}{}_k = \delta_{jk} \Rightarrow R^T R = 1$$
  In other words, rotation matrices are <u>orthogonal</u>
- We already have something like a scalar product — the invariant interval
  + If we consider δx^μ as a 4-vector,
  $$\delta s^2 = -(\delta x^0)^2 + (\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2$$
  + To write this with index notation, we need to introduce the <u>metric</u>
  $$\delta s^2 = \eta_{\mu\nu}\, \delta x^\mu \delta x^\nu \quad \text{where} \quad [\eta_{\mu\nu}] = \begin{array}{c|l} & \nu \rightarrow \\ \hline \mu \downarrow & \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \end{array}$$
  + Suppose you calculate δs^2 in another frame
  $$\delta s^2 = \eta_{\mu'\nu'}\, \delta x^{\mu'} \delta x^{\nu'} = (\eta_{\mu'\nu'} \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta)\, \underbrace{\delta x^\alpha \delta x^\beta}_{}\ \text{S frame}$$
  For this to be the same as in the S frame, we need
  $$\eta_{\mu'\nu'} \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta = \eta_{\alpha\beta} \quad \Rightarrow \quad \Lambda^T \eta \Lambda = \eta \qquad (\star)$$
  <u>Note:</u> In special relativity, the metric η is the same in every frame
  + We say Λ is a Lorentz transformation if it satisfies $(\star)$
  This includes 3D rotations $[\Lambda] = \begin{bmatrix} 1 & \\ & R \end{bmatrix}$, etc.
- This is a scalar product for any 2 4-vectors
  + Frame S: $\eta_{\mu\nu} a^\mu b^\nu = \eta_{\mu'\nu'} a^{\mu'} b^{\nu'}$; frame S' $= a \cdot b$
  + You can take the scalar product of a vector with itself to <u>square</u> it
  $a^2 > 0$ spacelike, $a^2 = 0$ lightlike, $a^2 < 0$ timelike
  classifies 4-vectors.