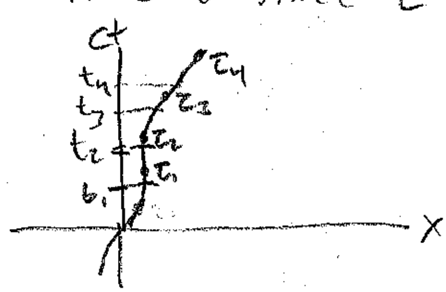
# Velocities and 4-Vectors

## Particle Velocities

- In any given reference frame, a particle follows a path $\vec{x}(t)$
  - Its velocity is therefore $\vec{u}(t) = \frac{d\vec{x}}{dt}(t)$
  - The invariant interval and proper time are still invariant for infinitesimal changes in space and time:
    + $$\delta s^2 = -c^2 \delta t^2 + \delta \vec{x}^2 \longrightarrow ds^2 = -c^2 dt^2 + d\vec{x}^2$$
      $$d\tau^2 = dt^2 - d\vec{x}^2/c^2$$
    + That's because the Lorentz transformations are linear so $dt' = \gamma(dt - v\,dx/c^2)$ etc
  - But then we have an instantaneous proper time for a moving particle
    $$d\tau = \sqrt{dt^2 - d\vec{x}^2(t)/c^2} = dt\sqrt{1 - \vec{u}(t)^2/c^2}$$
    + That's time dilation again
    + For changing motion, you can integrate this to get a total proper time
      $$\tau_f - \tau_i = \int_{t_i}^{t_f} dt \sqrt{1 - \vec{u}^2(t)/c^2}$$
    + That's the same in every frame (ie, $dt$ or $dt'$)
  - It can be convenient to label the particles world line by $\tau$, the proper time along the worldline, rather than a specific reference frame's time $t$ since $\tau$ is invariant



• Example (somewhat contrived to make the math work)

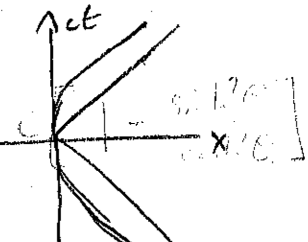
Consider a particle moving along the path $x(t) = ct_0 \ln(\cosh(t/t_0))$

+ Reminder: Hyperbolic trig. functions are

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}, \text{ etc}$$

$$\cosh^2\theta - \sinh^2\theta = 1 \;;\; \frac{d}{d\theta}(\sinh\theta) = \cosh\theta, \quad \frac{d}{d\theta}(\cosh\theta) = \sinh\theta$$

+ Plot the worldline:




$\cosh(0) = 1 \Rightarrow x(0) = 0$

At large $t$, $\cosh(t/t_0) \simeq e^{t/t_0}$

$x(t) \simeq ct_0 \ln(e^{t/t_0}) \simeq ct$ ← nearly lightspeed

+ The velocity is $u(t) = \frac{dx}{dt} = c\,\frac{d}{d\theta}\left(\ln(\cosh\theta)\right) = c\tanh(t/t_0)$ chain rule

$u(t=0) = 0, \quad u(t \to \pm\infty) = \pm c$

+ The particle's proper time is

$$\tau = \int dt\,\sqrt{1-u^2/c^2} = \int dt\,\sqrt{1-\tanh^2(t/t_0)} = \int \frac{dt}{\cosh(t/t_0)}$$

This is a bit of a complicated integral. The solution is

$$\tau = 2t_0 \tan^{-1}\left[\tanh(t/2t_0)\right] \quad \text{or} \quad \tan(\tau/2t_0) = \tanh(t/2t_0)$$

We chose to set $\tau = 0$ at $t = 0$   $\{\tanh(\infty) = 1, \ \tan(\pi/4) = 1$

Does this make sense? $\tau \to \pm\frac{\pi}{2}t_0$ as particle speed $\to c$, so that's good

At $t \simeq 0$, $\tan(\tau/2t_0) \simeq \tau/2t_0$ and $\tanh(t/2t_0) \simeq t/2t_0$, so $\tau \simeq t$. Also good.

+ Finally, we can reparameterize in terms of $\tau$

$$x(\tau) = ct_0 \ln\left[\cosh\left(2\tanh^{-1}(\tan(\tau/2t_0))\right)\right] = \text{double angle formulas} \ldots$$

$$= -ct_0 \ln\left[\cos(\tau/t_0)\right] \quad (\text{Goes to } \pm\infty \text{ at } \tau = \pm\tfrac{\pi}{2}t_0)$$

$t(\tau) = 2t_0 \tanh^{-1}\left(\tan(\tau/2t_0)\right)$; $u(\tau) = c\sin(\tau/t_0) = \frac{dx}{d\tau}\frac{d\tau}{dt} \neq \frac{dx}{d\tau}$

- The Lorentz Transformation for velocities

We want to see how velocities change from one inertial frame to another. Take frames $S$ and $S'$ in Standard configuration (relative motion along $x$)

- Remember, in Newtonian physics

$$\vec{u}' = \frac{d\vec{x}'}{dt} = \frac{d}{dt}(\vec{x} - \vec{v}t) = \frac{d\vec{x}}{dt} - \vec{v} = \vec{u} - \vec{v} \quad \text{simple}$$

- We take the same procedure here but remember $t' \neq t$ anymore
  + The Lorentz transformations will be handy

$$dt' = \gamma(dt - v\,dx/c^2); \; dx' = \gamma(dx - v\,dt), \; dy' = dy, \; dz' = dz.$$

  + Then we can look at each component of $\vec{u}'$ which does not have to be along $x$
  + The $x$-component

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - v\,dx/c^2)} = \frac{dx/dt - v}{1 - (v/c^2)\,dx/dt} = \frac{u_x - v}{1 - v u_x/c^2}$$

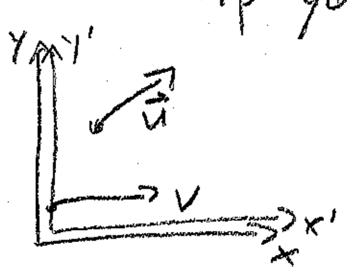
  + The $y$ and $z$ components are similar

$$u_{y,z}' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - v\,dx/c^2} = \frac{1}{\gamma(v)} \frac{dy/dt}{1 - \frac{v}{c^2}\,dx/dt} = \frac{1}{\gamma(v)} \frac{u_y}{1 - \frac{v u_x}{c^2}}$$

and

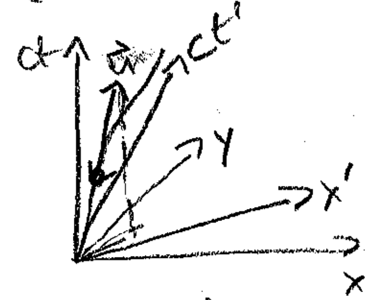
$$u_z' = \frac{u_z}{\gamma(v)} \frac{u_z}{1 - v u_x/c^2}$$

  + Pictures to help you visualize




what is $\vec{u}'$?

or




3D Spacetime diagram

- To summarize, the velocity transformations are

$$u_x' = \frac{u_x - v}{1 - v u_x/c^2}, \quad u_{y,z}' = \frac{1}{\gamma(v)} \frac{u_{y,z}}{1 - v u_x/c^2}$$

$$u_x = \frac{u_x' + v}{1 + v u_x'/c^2}, \quad u_{y,z} = \frac{1}{\gamma(v)} \frac{u_{y,z}'}{1 + v u_x/c^2}$$

These do not look like Lorentz transformations of coordinates

- Immediate consequences
  - You can't boost to higher than the speed $c$.

    As an example, imagine that $\vec{u}$ is along $x$ $(u_y = u_z = 0)$

    then $$u_x' = \frac{u_x - v}{1 - v u_x/c^2} \xrightarrow[v \to c]{} \frac{u_x - c}{1 - u_x/c} = -c$$

    Remember that $v \le c$ for any pair of frames

    Can you understand the sign?
  - If you have 2 particles with velocities $\vec{u}_1, \vec{u}_2$, these let you find the relative velocity (the velocity of particle 2 in the rest frame of particle 1 or vice versa) — Called "velocity addition"
  - If you boost twice (both times in the $x$ direction), you have another boost. (Boost first by $v_1$ then by $v_2$:

    Go back to normal coordinate Lorentz transformations

$$x'' = \gamma_2 (x' + v_2 t'), \quad t'' = \gamma_2 (t' + v_2 x'/c^2)$$

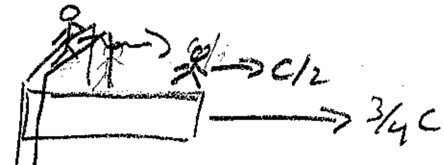
$$x'' = \gamma_2 \gamma_1 \left[ (x - v_1 t) - v_2 (t - v_1 x/c^2) \right] = \gamma_2 \gamma_1 \left[ \left(1 + \frac{v_1 v_2}{c^2}\right) x - (v_1 + v_2) t \right]$$

etc.

The combined boost must be velocity

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}$$ (look familiar?). Can you see $\gamma_3 = \gamma_1 \gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right)$?

- A quick example or two.
  - A pelativistic train moves at speed $\frac{3}{4}c$.
    A thief has stolen the Hope Diamond and escapes through the roof. He activates a jet pack capable of flying at $c/2$ (relative to the train).




The police arrie at an overpass just after the train has passed and shoot at the thief with a laser. What happens?

1) Relative to the earth, the thief moves at velocity (train is frame S', earth is frame S)

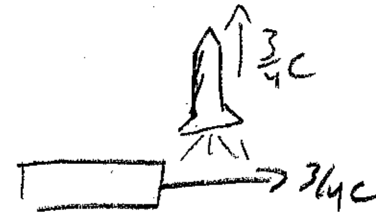
$$u_x = \frac{u_x' + v}{1 + v u_x'/c^2} = \frac{c/2 + 3c/4}{1 + (3/4)(1/2)} = \frac{5/4\, c}{11/8} = \frac{10}{11}c.$$

So the laser beam blasts his jet pack.

2) Relative to the train, the thief moves at $c/2$ and the laser beam at $c$, so he still gets hit

3) Relative to the thief, the police + train move backward, but the laser beam still moves at speed $c$!

- We have the same relativistic train moving past a launchpad where a rocket has just launched at $\frac{3}{4}c$.




What's the rocket speed relative to the train?

$$u_x' = \frac{u_x - v}{1 - v u_x/c^2} = -v = -\frac{3}{4}c, \quad u_z' = \frac{u_z}{\gamma(v)\left(1 - \frac{v u_x}{c^2}\right)} = \sqrt{1 - \left(\tfrac{3}{4}\right)^2}\left(\tfrac{3}{4}c\right)$$

$$= \frac{3\sqrt{7}}{16}c < \frac{3}{4}c$$

Total speed

$$\sqrt{(u_x')^2 + (u_z')^2} = \sqrt{\frac{9}{16} + \frac{63}{16^2}}\,c = \frac{3\sqrt{23}}{16}c \approx \frac{7}{8}c$$