Momentum + Angular Momentum Wavefunctions

— Remember that only <u>eigenfunctions</u> of the momentum operator describe particles with a definite momentum

- Momentum operator is a derivative $\vec{p} = -i\hbar\vec{\nabla}$ or $p = -i\hbar\frac{d}{dx}$
- An eigenfunction $\psi_{\vec{p}}$ satisfies $\vec{p}\cdot\psi_{\vec{p}} = -i\hbar\vec{\nabla}\psi_p(\vec{x}) = \vec{p}\,\psi_{\vec{p}}(\vec{x})$

  operator action multiplication

  + In 1D, this is $-i\hbar\frac{d\psi_p(x)}{dx} = p\,\psi_p(x) \Rightarrow \psi_p(x) \propto e^{ipx/\hbar}$
  + In 3D, it's just a product! $\psi_{\vec{p}}(\vec{x}) \propto e^{ip_x x/\hbar}e^{ip_y y/\hbar}e^{ip_z z/\hbar}$ $= e^{i\vec{p}\cdot\vec{x}/\hbar}$
  + The <u>eigenvalue</u> p (or $\vec{p}$) is the definite momentum
- The allowed values of momentum depend on boundary conditions
  + For periodic b.c., $\psi(x) = \psi(x+2\pi R)$, we need $p = \hbar n/R$ for n integer. Then $\psi_p(x) = \frac{1}{\sqrt{2\pi R}}e^{inx/R}$ (normalized)
  + For Dirichlet or Neumann b.c., momentum eigenfunctions are not allowed!
  + For the whole real line $-\infty < x < \infty$, any p is allowed, but ψ_p is not normalizable! We still write $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$

— But you can actually write <u>any</u> wavefunction in terms of momentum eigenfunctions

- Remember the <u>Fourier series</u> for any periodic function:
  + If $\psi(x) = \psi(x+2\pi R)$, $\psi(x) = \sum_{n=-\infty}^{\infty}\frac{c_n}{\sqrt{2\pi R}}e^{inx/R} = \sum_n c_n\psi_{p=\hbar n/R}(x)$
    $= \sum_n c_n \psi_{p=\hbar n/R}(x)$

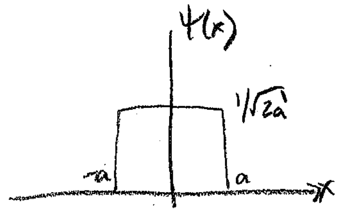
+ The Fourier coefficient is $c_n = \frac{1}{\sqrt{2\pi R}} \int_0^{2\pi R} dx\, e^{-inx/R}\, \psi(x)$
$= \int dx\, \psi_{p_n}^*(x)\, \psi(x)$

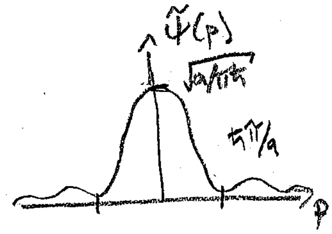
+ These coefficients hold all the same information as the wavefunction! It all relates to inner products + linear algebra

- On the full real line, we have the Fourier transform
  + $$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \phi(k)\, e^{ikx} = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, \tilde{\psi}(p)\, \psi_p(x)$$
    where $p = \hbar k$ and $\tilde{\psi}(p=\hbar k) = \phi(k)/\sqrt{\hbar}$
  + We call $\phi(k)$ the Fourier transform and $\tilde{\psi}(p)$ the momentum space wavefunction
  + The inverse transform gives $\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx\, \psi_p^*(x)\, \psi(x)$
  + Doing the F.T. followed by inverse F.T. shows
    $$\int \frac{dk}{2\pi}\, e^{ik(x-x')} = \delta(x-x')$$
    See reading + HW for more on the Dirac delta function

- Properties of the momentum space wavefunction
  + Just like $x \cdot \psi(x) = x\,\psi(x)$ (ie, operator action = multiplication)
    $p \cdot \tilde{\psi}(p) = p\,\tilde{\psi}(p)$
  + But conversely, $x \cdot \tilde{\psi}(p) = i\hbar \frac{d\tilde{\psi}}{dp}(p)$ (you will prove)
  + If $\psi(x)$ is normalized, so is $\tilde{\psi}(p)$: $\int_{-\infty}^{\infty} dp\, |\tilde{\psi}(p)|^2 = 1$
  + This is just as "valid" of a wavefunction as the usual one.

- There is an inverse relationship between the spread of a wavefunction in x and in p. Take the example $\psi(x) = \begin{cases} 1/\sqrt{2a} & \text{for } |x|<a \\ 0 & |x|>a \end{cases}$
  + The momentum space wavefunction is
$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{a} dx\, e^{-ipx/\hbar} \left(\frac{1}{\sqrt{2a}}\right) = \cdots = \sqrt{\frac{\hbar}{\pi a}}\, \frac{\sin(pa/\hbar)}{p}$$
  + $\tilde{\psi}(p)$ first vanishes at $p = \pm \hbar\pi/a$, which explains how wide the wavefunction is





  + The wider in x, the narrower in p, and vice-versa. This holds in general.
  + Note that $\Delta x = a$ and $\Delta p = \hbar\pi/a$ satisfy $\Delta x \Delta p = \hbar\pi > \hbar/2$. This is the heuristic reason for the <u>Heisenberg uncertainty relation</u> $\Delta x \Delta p \geq \hbar/2$.

- This is of course all related to your discussion of <u>wave packets</u> (see last term + Griffiths).

- Angular Momentum Eigenfunctions

- Similarly, particles with definite angular momentum have wavefunctions that are eigenfunctions of $\vec{L}$ operators
  + Actually, we can only have eigenfunctions of L_z and $\vec{L}^2$ simultaneously, not L_x, L_y, and L_z
  + The eigenfunctions of L_z + $\vec{L}^2$ are called <u>spherical</u> <u>harmonics</u> Y_ℓ^m
  + $L_z = -i\hbar \frac{\partial}{\partial\phi}$ and $\vec{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$
    (angular part of Laplacian)
  + This means $Y_\ell^m(\theta,\phi) = f(\theta)\, e^{im\phi}$
    Since ϕ has periodicity 2π, m = <u>integer</u>, $L_z \cdot Y_\ell^m = m\hbar\, Y_\ell^m$
  + The equation $\vec{L}^2 \cdot Y_\ell^m = \hbar^2 \ell(\ell+1)\, Y_\ell^m$ is actually known as a D.E.
    $f(\theta)$ is a special type of function called an "associated Legendre function." Non singular <u>only</u> for $\ell = 0, 1, \cdots$, $m = -\ell, -\ell+1, \ldots, \ell$

- Any function of angles on a sphere can be written as a sum of spherical harmonics (like Fourier series)
  + $g(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_\ell^m(\theta,\phi)$ uniquely
  + Again, $a_{\ell m}$ coefficients contain <u>all</u> the information
  + The coefficients are

$$a_{\ell m} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(Y_\ell^m(\theta,\phi)\right)^* g(\theta,\phi)$$

  + This is used in quantum mechanics, electromagnetic theory, astronomy, ...
  + You can also see

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(Y_{\ell'}^{m'}(\theta,\phi)\right)^* Y_\ell^m(\theta,\phi) = \delta_{\ell,\ell'}\,\delta_{m,m'}$$