PHYS-4601 Homework 8 Due 17 Nov 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Isotropic Harmonic Oscillator** *from Griffiths 4.38,39*

   Consider a harmonic oscillator where the restoring force is independent of the direction. In this case, the potential is

   $$V(r) = \frac{1}{2} m\omega^2 r^2 \ . \tag{1}$$

   (a) Show that the energy eigenvalues are $E_n = \hbar\omega(n + 3/2)$, where n is any non-negative integer. It's easiest to do this using separation of variables in Cartesian coordinates.

   (b) Find the degeneracy of states with energy E_n.

   (c) Now consider the Schrödinger equation in spherical coordinates and restrict to the case $\ell = 0$ for the radial equation. We know that the ground state has $\psi \propto \exp[-\rho^2/2]$ from separation of variables in Cartesian coordinates, where $\rho = \sqrt{m\omega/\hbar}\, r$. Therefore, we define $u = v(\rho)\exp[-\rho^2/2]$, where $v(\rho) = \rho + \cdots$ is a polynomial. Argue that the (unnormalized) wavefunctions $u = H_n(\rho)\exp[-\rho^2/2]$ solve the radial equation for any odd n. Find the associated energies for $n = 1, 3$. Did you find the energies you expected from part (a)? Explain why or why not.

2. **Spherical Wells** *extended from Griffiths 4.8,4.9*

   (a) First, consider the $\ell = 1$ states of the infinite spherical well ($V(r) = 0$ for $r < a$, $V(r) = \infty$ for $r > a$). Show that the allowed energy eigenvalues are $E_n = \hbar^2 x_n^2/2ma^2$, where x_n is the nth root of the spherical Bessel function $j_1(x)$, and that $E_n \approx (\hbar^2\pi^2/2ma^2)(n + 1/2)^2$ for $n \gg 1$. *Hint:* for the last part, you will need to know an expression of $j_1(x)$ in terms of trig functions. By definition $x_1 < x_2 < \cdots$.

   In the rest of this problem, donsider a particle of mass m in the spherically symmetric potential

   $$V(r) = \begin{cases} -V_0 & (r < a) \\ 0 & (r \geq a) \end{cases} \ . \tag{2}$$

   In 1D quantum mechanics, any potential that goes to zero at infinity and is negative anywhere has at least one bound state. We will see that is not true in 3D.

   (b) Assume $\ell = 0$ and energy $E < 0$. Find a transcendental equation that determines E. What is the condition on V_0 that allows a bound state?

   (c) Use Maple to solve the transcendental equation of part (b) and plot the ground state energies as a function of V_0 in the range $\pi^2\hbar^2/8ma^2 < V_0 < \pi^2\hbar^2/2ma^2$. Attach a printout of your code and the plot. *Hint:* One way to proceed is to solve for the energy with a particular value of V_0 using `fsolve` and step through values of V_0 using `seq`, then `listplot`.

3. **Electromagnetic Gauge Transformations** *Griffiths 4.61*

   Now that we're in 3D, we could imagine having an electromagnetic field. For a particle of charge q in potential Φ and vector potential $\vec{A}$, the Hamiltonian is

   $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi \ . \tag{3}$$

The electric and magnetic field are

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \tag{4}$$

For more details, see Griffiths problem 4.59.

(a) Show that the electromagnetic fields are invariant under *gauge transformations*. That is, show that the potentials

$$\Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \ , \quad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{5}$$

give the same $\vec{E}$ and $\vec{B}$ fields as Φ and $\vec{A}$, where Λ is any function of $\vec{x}$ and t.

(b) Since the Hamiltonian involves the potentials, it looks like we can't just make a gauge transformation in the quantum theory. However, assuming that a wavefunction $\Psi(\vec{x}, t)$ solves the time-dependent Schrödinger equation for potentials Φ and $\vec{A}$, show that

$$\Psi' = e^{iq\Lambda/\hbar}\Psi \tag{6}$$

solves the time-dependent Schrödinger equation for the potentials Φ' and $\vec{A}'$ given in (5).

This gauge invariance is a critical feature of the quantum mechanical theory of electromagnetism with profound consequences. We may explore aspects of it again in assignments.